

A Note on the Theorem of Fatou-Pólya

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Received January 31, 1964

1. Theorem 1. *Es sei vorgelegt eine Potenzreihe mit endlichem Konvergenzradius. Durch die Abänderung der Vorzeichen gewisser geeignet gewählter Koeffizienten entsteht daraus eine Potenzreihe, für welche alle Punkte des Konvergenzkreises singuläre Punkte sind.*

This theorem was conjectured by Fatou, and first proved by Pólya. The purpose of the present paper is to give another proof. The proof of Theorem 1. will be based on the following lemmas. We can assume, without loss of generality, that the series has radius of convergence 1.

Lemma 1. ([1]) If a series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ has radius of convergence 1, then in order that $z = e^{i\theta}$ is a singular point it is necessary and sufficient that

$$(1) \quad \lim_{n \rightarrow \infty} \left| \sqrt[n]{\sum_{m=0}^n \binom{n}{m} c_m e^{im\theta}} \right| = 2.$$

Lemma 2. ([1]) If a series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence 1,

$$(2) \quad \lim_{n \rightarrow \infty} \left| \sqrt[n]{\sum_{m=0}^n \binom{n}{m} c_m e^{im\theta}} \right| \leq 2,$$

for an arbitrary real number θ ($0 \leq \theta < 2\pi$).

Lemma 3. ([3]) When $0 < \theta < 1$ and n is sufficiently large, there exists a number r (< 1) which the following inequality holds.

$$(3) \quad \binom{2n}{[\theta n]} < r^n \cdot 2^{2n},$$

where $[\]$ is the Gaussian symbol, r is independent on n , and only depend on θ .

Let $\sum_{n=0}^{\infty} c_n z^n$ be a given power series with radius of convergence 1. In 2. we shall show a practical process to form a sequence $\{\epsilon_n\}$ ($\epsilon_n = 1$ or -1) such that the unit circle is a natural boundary of $\sum_{n=0}^{\infty} \epsilon_n c_n z^n$. Next

in 3. we shall prove that the process in 2. is correct.

2. (i) Since the series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence 1, for every positive value of ϵ , no matter how small, an infinite number of terms of the sequence can be found such that

$$(4) \quad (1-\epsilon)^n < |c_n| < (1+\epsilon)^n.$$

From the sequence $\{c_n\}$ we are going to extract a subsequence $\{c_{n_k}\}$ which satisfies (4).

(ii) Further, from the sequence $\{n_k\}$ we shall extract an arbitrary subsequence $\{n_{k_j}\}$ which satisfies the following condition.

Condition. For a positive number θ (< 1),

$$(5) \quad (2-\theta)n_{k,j-1} < \theta n_{k,j} \text{ for } j=1, 2, 3, \dots$$

For convenience we shall replace $\{n_{k,j}\}$ by $\{n_k\}$. Consequently

$$(6) \quad (2-\theta)n_{k-1} < \theta n_k \text{ for } k=1, 2, 3, \dots$$

Note 1. It is clear that

$$(7) \quad 2n_{k-1} - [\theta n_{k-1}] - 1 < [\theta n_k] + 1.$$

(iii) Let the 2^m th roots of unity denote by $1, \omega_m, \omega_m^2, \dots, \omega_m^{2^m-1}$. In good order we arrange numbers given by taking $m=1, 2, 3, \dots$, and we form the sequence

$$(8) \quad 1, -1, 1, i, -1, -i, 1, \omega_3, \omega_3^2, \dots, \omega_3^7, \dots$$

Note 2. For an arbitrary positive integer N , each of 2^N th roots of unity appears infinite times in this sequence. To the k th term ω_m^s ($0 \leq s \leq 2^m-1$) of this sequence, n_k makes to correspond. It is as follows.

$$(9) \quad \begin{matrix} n_1, n_2, n_3, \dots, n_k, \dots \\ 1, -1, 1, \dots, \omega_m^s, \dots \end{matrix}$$

$$(iv) \quad \begin{aligned} & \text{Since } |c_{n_k}(\omega_m^s)^{n_k}| = |c_{n_k}| > (1-\epsilon)^{n_k}, \\ & |\alpha_{n_k}| + |\beta_{n_k}| > (1-\epsilon)^{n_k}, \end{aligned}$$

where $c_{n_k}(\omega_m^s)^{n_k} = \alpha_{n_k} + i\beta_{n_k}$.

Accordingly either $|\alpha_{n_k}|$ or $|\beta_{n_k}|$ is greater

than $\frac{1}{2}(1-\varepsilon)^{n_k}$.

For l satisfying

$$(10) \quad [\theta n_k] + 1 \leq l \leq 2n_k - [\theta n_k] - 1,$$

(a) When $|\alpha_{n_k}| > \frac{1}{2}(1-\varepsilon)^{n_k}$, we find ε_l such that

$$(11) \quad \operatorname{Re}\{\varepsilon_l c_l(\omega_m^s)^l\} \cdot \operatorname{Re}\{c_{n_k}(\omega_m^s)^{n_k}\} \geq 0.$$

(b) When $|\beta_{n_k}| > \frac{1}{2}(1-\varepsilon)^{n_k}$, we find ε_l such that

$$(12) \quad I_m\{\varepsilon_l c_l(\omega_m^s)^l\} \cdot I_m\{c_{n_k}(\omega_m^s)^{n_k}\} \geq 0.$$

(c) When $|\alpha_{n_k}|$ and $|\beta_{n_k}|$ are both greater than $\frac{1}{2}(1-\varepsilon)^{n_k}$, we may choose either (a) or (b).

(v) Every ε_l which is not in the above discussion may be chosen freely.

3. For an arbitrary positive integer m , let one of 2^m th roots of unity denote by ω_m^s , then $z = \omega_m^s$ is a singular point. The proof is given as under.

There is an infinite number of terms of the sequence $\{n_k\}$ which correspond to ω_m^s . (Note 2.)

Let n_k be e.g. one of them, then the following inequality holds.

$$(13) \quad \left| \sum_{l=1}^{2n_k} \binom{2n_k}{l} \varepsilon_l c_l(\omega_m^s)^l \right| \geq \left| \sum_{l=[\theta n_k]+1}^{2n_k - [\theta n_k] - 1} \binom{2n_k}{l} \varepsilon_l c_l(\omega_m^s)^l \right| \\ - \left| \sum_{l=2n_k - [\theta n_k]}^{[\theta n_k]} \binom{2n_k}{l} \varepsilon_l c_l(\omega_m^s)^l \right|.$$

For sufficiently small ε and large n_k ,

(i) when $|\alpha_{n_k}| > \frac{1}{2}(1-\varepsilon)^{n_k}$, by the pro-

cess in 2., lemma 3. and Wallis's formula,

$$(14) \quad \left| \sum_{l=1}^{2n_k} \binom{2n_k}{l} \varepsilon_l c_l(\omega_m^s)^l \right| > \left| \sum_{l=[\theta n_k]+1}^{2n_k - [\theta n_k] - 1} \binom{2n_k}{l} \varepsilon_l c_l(\omega_m^s)^l \right| \\ \operatorname{Re} \varepsilon_l c_l(\omega_m^s)^l - r_1^{n_k} \cdot 2^{2n_k} \geq \binom{2n_k}{n_k} |\alpha_{n_k}| \\ - r_1^{n_k} \cdot 2^{2n_k} > |\alpha_{n_k}| \frac{2^{2n_k}}{\sqrt{n_k}} \left(\frac{1}{\sqrt{\pi}} - \varepsilon \right) - r_1^{n_k} \cdot 2^{2n_k} \\ > 2^{2n_k} (1-\varepsilon)^{2n_k},$$

where r_1 is smaller than 1, and only depend on θ .

(ii) When $|\beta_{n_k}| > \frac{1}{2}(1-\varepsilon)^{n_k}$, by similar discussion in (i), (14) holds. Since there is an infinite number of terms of the sequence $\{n_k\}$ which correspond to ω_m^s ,

$$(15) \quad \lim_{n \rightarrow \infty} \left| \sqrt{\sum_{l=0}^n \binom{n}{l} \varepsilon_l c_l(\omega_m^s)^l} \right| \geq 2.$$

From lemma 2.

$$(16) \quad \lim_{n \rightarrow \infty} \left| \sqrt{\sum_{l=0}^n \binom{n}{l} \varepsilon_l c_l(\omega_m^s)^l} \right| = 2.$$

Consequently $z = \omega_m^s$ is a singular point.

Since m is an arbitrary positive integer, the unit circle is a natural boundary. q.e.d.

References

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