

ON THE CONFORMAL MAPPING OF NEARLY CIRCULAR DOMAINS

By

Hideo SAKASHITA

Received September 30, 1955

Introduction

While investigating the paper of Herr Müller, we found the theorem which states as follows:

Das einfach zusammenhängende Gebiet G_z liege in Kreis $|z| < 1$ und enthalte den Kreis $|z| < p$; das einfach zusammenhängende Gebiet G_w liege in Kreis $|w| < 1$ und enthalte den Kreis $|w| < q$. Die Funktion $w=f(z)$ bilde G_z auf G_w derart konform und schlicht ab, daß $f(0)=0$, $f'(0)>0$. Dann gilt für $|z| \leq \theta p$ ($0 \leq \theta < 1$) die Abschätzung

(1) $|f(z) - z| \leq \theta \sqrt{(pR)^2 + 4p \sin^2 A(p, \theta)}$,
wobei

(2) $R = \text{Max} \left\{ \frac{1}{p} - 1, 1 - q \right\}$,

(3) $A(p, \theta) = \text{Min} \left\{ \frac{\pi}{2}, \frac{1}{\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta} \right\}$,
in der Ungleichung (1) gilt für $\theta > 0$ das Gleichheitszeichen nur, wenn $p=1$, $f(z)=qz$, $|z|=\theta$ ist, dann aber für all θ aus dem Intervall $0 < \theta < 1$.

is not always true. Therefore we endeavored to correct the error, and fortunately we succeeded to obtain a new result which is sharper than that of Ostrowski.

I

Let $P[w]$ denote a point of the complex plane, whose affix is w , and G_w a simply-connected domain whose boundary is consisted of the following arcs:

The major arc $\left(P\left[\frac{1}{2}e^{\frac{17}{12}\pi i}\right], P\left[\frac{1}{2}e^{\frac{19}{12}\pi i}\right] \right)$

of the circle $|w| = \frac{1}{2}$, the segment

$\left(P\left[\frac{1}{2}e^{\frac{19}{12}\pi i}\right], P\left[\frac{2}{3}e^{\frac{19}{12}\pi i}\right] \right)$, the minor arc

$\left(P\left[\frac{2}{3}e^{\frac{19}{12}\pi i}\right], P\left[\frac{2}{3}\right] \right)$ of the circle $|w| = \frac{2}{3}$,

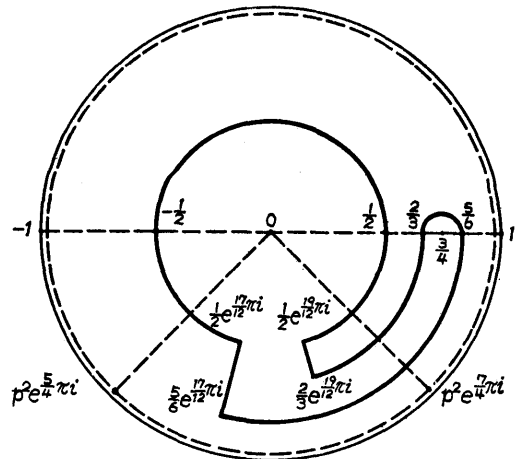
the semicircle $\left| w - \frac{3}{4} \right| = \frac{1}{12}$, $\text{Im}\{w\} > 0$,

the minor arc $\left(P\left[\frac{5}{6}\right], P\left[\frac{5}{6}e^{\frac{17}{12}\pi i}\right] \right)$ of the

circle $|w| = \frac{5}{6}$, and the segment

$\left(P\left[\frac{5}{6}e^{\frac{17}{12}\pi i}\right], P\left[\frac{1}{2}e^{\frac{17}{12}\pi i}\right] \right)$.

We put $q = \frac{1}{2}$.



Let the function $w=f_r(z)$ ($f_r(0)=0$, $f'_r(0)>0$) be regular and univalent and maps $|z| < r$ conformally onto the interior of G_w , and let $z=\varphi_r(w)$ be the inverse function of $w=f_r(z)$.

Being $\varphi_1(w) = \frac{1}{r} \varphi_r(w)$, $\left| \varphi_1\left(\frac{3}{4}\right) \right| < 1$, if we take r_1 which satisfies $\left| \varphi_1\left(\frac{3}{4}\right) \right| < r_1 < 1$, then we have

$$r_1^2 > r_1 \left| \varphi_1\left(\frac{3}{4}\right) \right| = r_1 \frac{1}{r_1} \left| \varphi_{r_1}\left(\frac{3}{4}\right) \right| = \left| \varphi_{r_1}\left(\frac{3}{4}\right) \right|.$$

Hence the image domain of $|z| < r_1^2$

given by $w=f_{r_1}(z)$ contains $P\left[\frac{3}{4}\right]$, i.e, the image curve C_{r_1} of $|z|=r_1$ intersects the segment $\left(P\left[\frac{2}{3}\right], P\left[\frac{5}{6}\right]\right)$ at more than one point. Put

$$\text{Max}\left(\frac{\varphi_1\left(\frac{3}{4}\right)+1}{2}, 0.99\right)=p<1, \theta=p \text{ and}$$

$$f(z)\equiv f_p(z),$$

further we denote the disk $|z|<p$ by G_p .

Then under the above mentioned conditions we can show that the following inequality does not hold.

$$(4) \quad |f(z)-z| \leq \theta \sqrt{(pR)^2+4p\sin^2 A} \\ = p \sqrt{\frac{1}{4}p^2+4p\sin^2 A} \text{ for } |z|=p^2.$$

Being

$$\frac{1}{\pi} \log \frac{1}{p} \log \frac{1+p}{1-p} \leq \frac{1}{\pi} \log_{0.99} \log_{0.01} 1.99 < 0.02,$$

it will be sufficient to show that the inequality

$$(5) \quad |f(z)-z| < \sqrt{\frac{1}{4}+0.0016} < 0.51$$

does not hold.

As the image domain of $|z|<p^2$ given by $w=f(z)$ contains $P[0]$ and $P\left[\frac{3}{4}\right]$, image curve C_{p^2} of $|z|=p^2$ intersects the segment $\left(P\left[-\frac{1}{2}\right], P\left[\frac{1}{2}\right]\right)$, the minor arc $\left(P\left[\frac{1}{2}e^{\frac{17}{12}\pi i}\right], P\left[\frac{1}{2}e^{\frac{19}{12}\pi i}\right]\right)$ of the circle $|w|=\frac{1}{2}$, and the segment $\left(P\left[\frac{2}{3}\right], P\left[\frac{5}{6}\right]\right)$ respectively.

Let $P[\alpha]$ be one of points of intersection with the former segment, $P[\beta_i](i=1, 2, \dots, n)$ be all points of intersection with the minor arc and $P[\gamma]$ be one of points of intersection with the latter segment.

On the other hand, being

$$\left(P\left[\frac{1}{2}e^{\frac{19}{12}\pi i}\right], P\left[p^2e^{\frac{7}{4}\pi i}\right]\right)^2 = (p^2)^2 + \left(\frac{1}{2}\right)^2 \\ - 2(p^2)\left(\frac{1}{2}\right)\cos \frac{\pi}{6} > 0.35,$$

we have

$$\left(P\left[\frac{1}{2}e^{\frac{19}{12}\pi i}\right], P\left[p^2e^{\frac{7}{4}\pi i}\right]\right) > \sqrt{0.35} > 0.51.$$

Therefore if we assume that (5) holds, (considering $|z|=p^2$ as $|w|=p^2$) all inverse image points of $P[\beta_i](i=1, 2, \dots, n)$ must be on the minor arc $(P[p^2e^{\frac{5}{4}\pi i}], P[p^2e^{\frac{7}{4}\pi i}])$ of the circle $|w|=p^2$. As the distance between the segment $(P[1], P[-1])$ and the

minor arc $(P[p^2e^{\frac{5}{4}\pi i}], P[p^2e^{\frac{7}{4}\pi i}])$ is $p^2 \sin \frac{\pi}{4}$, and

$$p^2 \sin \frac{\pi}{4} \geq 0.99 \times \frac{\sqrt{2}}{2} > 0.51,$$

the inverse image points of $P[\alpha]$ and $P[\gamma]$ must be on the major arc $(P[p^2e^{\frac{5}{4}\pi i}], P[p^2e^{\frac{7}{4}\pi i}])$ of the circle $|w|=p^2$.

Therefore the image curve of the arc $(P[p^2e^{ik_1}], P[p^2e^{ik_2}])$, where $P[p^2e^{ik_1}]$ and $P[p^2e^{ik_2}]$ are the inverse image points of $P[\alpha]$ and $P[\gamma]$, respectively, which is a part of the major arc $(P[p^2e^{\frac{5}{4}\pi i}], P[p^2e^{\frac{7}{4}\pi i}])$ of the circle $|w|=p^2$, is a Jordan curve which joins $P[\alpha]$ and $P[\gamma]$, therefore it contains at least one of $P[\beta_i](i=1, 2, \dots, n)$.

This contradicts the fact that there is a one-to-one continuous correspondence between the points of $|w|=p^2$ and those of C_{p^2} .

II

Theorem. On the same assumption as Müller's we have

$$(6) \quad |f(z)-z| \leq \theta \sqrt{(pR)^2+4p\sin^2 A(p, q, \theta)},$$

where

$$(7) \quad R = \text{Max}\left(\frac{1}{p}-1, 1-q\right),$$

$$(8) \quad A(p, q, \theta) = \text{Min}\left(\frac{\pi}{2}, \frac{1}{2\pi} \log \frac{1}{pq} \log \frac{1+\theta}{1-\theta}\right).$$

Moreover the equality holds when and only when

$$p=q=1, f(z)\equiv z \text{ for } 1>\theta>0.$$

Proof. Since $f(0)=0$ and $|f(z)|<1$ for $|z|<1$, by using the Schwarz's lemma we have

$$(9) \quad \left|\frac{f(z)}{z}\right| \leq \frac{1}{p} \text{ for } |z|<p.$$

It is evident that (9) is valid in the part of G_p which belongs to the ring domain $p \leq |z| < 1$.

Hence (9) holds in the whole G_p .

Similarly for the inverse function $z=\varphi(w)$ of $w=f(z)$ we can see easily that the following inequality holds.

$$\left|\frac{\varphi(w)}{w}\right| \leq \frac{1}{q} \quad w \in G_w.$$

Hence

$$(10) \quad \left|\frac{z}{f(z)}\right| \leq \frac{1}{q} \text{ in } G_z.$$

If we put

$$(11) \quad \left|\frac{f(z)}{z}\right| = \rho(z), \arg \frac{f(z)}{z} = \alpha(z),$$

by (9) and (10) we get

$$(12) \quad 0 < q \leq \rho(z) \leq \frac{1}{p} \text{ in } G_z,$$

$$(13) \quad |\rho(z) - 1| \leq \max\left\{\frac{1}{p} - 1, 1 - q\right\} = R$$

$$(14) \quad |f(z) - z| = |z| \left| \frac{f(z)}{z} - 1 \right| \\ = |z| \left| \rho(z) e^{i\alpha} - 1 \right| \\ = |z| \left\{ (1 - \rho(z))^2 + 4\rho(z) \sin^2 \frac{\alpha(z)}{2} \right\}^{\frac{1}{2}}.$$

By (11) and (12) we have $\frac{f(z)}{z} \neq 0$ in G_z , therefore if we put $z = re^{i\theta}$, $r < k < p$, and we apply the Poisson's formula to $\arg \frac{f(z)}{z}$ where we take the branch such that $\left(\arg \frac{f(z)}{z}\right)_{z=0} = \arg f(0) = 0$, we have

$$(15) \quad \alpha(z) = \alpha(re^{i\theta}) \\ = \frac{1}{\pi} \int_0^{2\pi} \log \rho(ke^{i\lambda}) \frac{rk \sin(\theta - \lambda)}{r^2 + k^2 - 2rk \cos(\theta - \lambda)} d\lambda \\ = \frac{1}{\pi} \int_0^{2\pi} \left\{ \log q + \log \frac{\rho(ke^{i\lambda})}{q} \right\} \\ \times \frac{rk \sin(\theta - \lambda)}{r^2 + k^2 - 2rk \cos(\theta - \lambda)} d\lambda \\ = \frac{1}{\pi} \int_0^{2\pi} \left\{ \log \frac{\rho(ke^{i\lambda})}{q} \right\} \\ \times \frac{rk \sin(\theta - \lambda)}{r^2 + k^2 - 2rk \cos(\theta - \lambda)} d\lambda.$$

Here we put

$$\log \left\{ \frac{\rho(ke^{i\lambda})}{q} \right\} \frac{rk \sin(\theta - \lambda)}{r^2 + k^2 - 2rk \cos(\theta - \lambda)} = K,$$

then as $\log \frac{\rho(ke^{i\lambda})}{q} \geq 0$ by (12), we have for $0 \leq \theta < \pi$

$$\int_0^\theta K d\lambda \geq 0, \int_\theta^{\pi+\theta} K d\lambda \leq 0, \int_{\pi+\theta}^{2\pi} K d\lambda \geq 0.$$

Therefore we have

$$\frac{1}{\pi} \log \frac{1}{pq} \int_0^{\pi+\theta} \frac{rk \sin(\theta - \lambda)}{r^2 + k^2 - 2rk \cos(\theta - \lambda)} d\lambda \leq \alpha(z) \\ \leq \frac{1}{\pi} \log \frac{1}{pq} \left\{ \int_0^\theta \frac{rk \sin(\theta - \lambda)}{r^2 + k^2 - 2rk \cos(\theta - \lambda)} d\lambda \right. \\ \left. + \int_{\pi+\theta}^{2\pi} \frac{rk \sin(\theta - \lambda)}{r^2 + k^2 - 2rk \cos(\theta - \lambda)} d\lambda \right\},$$

that is,

$$(16) \quad -\frac{1}{\pi} \log \frac{1}{pq} \log \frac{k+r}{k-r} \leq \alpha(z) \\ \leq \frac{1}{\pi} \log \frac{1}{pq} \log \frac{k+r}{k-r}.$$

By similar discussion we have (16) when $\pi \leq \theta < 2\pi$.

As the inequality (16) holds for $r < k < p$, it therefore must hold for $k \rightarrow p$, consequently the inequality

$$(17) \quad |\alpha(z)| \leq \frac{1}{\pi} \log \frac{1}{pq} \log \frac{p+r}{p-r}$$

holds.

If we consider $|z| = r \leq \theta p$, we have

$$(18) \quad |\alpha(z)| \leq \frac{1}{\pi} \log \frac{1}{pq} \log \frac{1+\theta}{1-\theta}.$$

From this we can conclude that

$$(19) \quad \sin^2 \frac{\alpha(z)}{2} = \sin^2 \frac{|\alpha(z)|}{2} \leq \sin^2 A(p, q, \theta).$$

By (12), (13), (14), and (19), we have

$$|f(z) - z| \leq \theta p \sqrt{R^2 + \frac{4}{p} \sin^2 A(p, q, \theta)} \\ = \theta \sqrt{(pR)^2 + 4p \sin^2 A(p, q, \theta)}.$$

By (9), (10), and $f'(0) > 0$, we can see that the equality sign does not occur except when $f(z) = \frac{1}{p}z$ or $f(z) = qz$, therefore by (13) we have

$$(20) \quad f(z) = \frac{1}{p}z, \quad R = \frac{1}{p} - 1; \quad f(z) = qz, \\ R = 1 - q.$$

These satisfy $\arg \frac{f(z)}{z} = 0$ respectively, accordingly for positive θ the equality sign occurs only when $p = q = 1$ by (18). In this case, of course, $f(z) = z$. This completes the proof.

III

Consider specially the case where $q = 1$, then (6) becomes

$$(21) \quad |f(z) - z| \leq \theta \sqrt{(1-p)^2 + 4p \sin^2 A(p, \theta)} \\ \text{for } |z| \leq \theta p,$$

where

$$A(p, \theta) = \min\left(\frac{\pi}{2}, \frac{1}{2\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta}\right).$$

To show that (21) is sharper than Ostrowski's result:

$$(22) \quad |f(z) - z| \leq \theta(1-p) + \theta \sqrt{1-p^2} \sqrt{\log \frac{1}{1-\theta^2}}$$

for $0 \leq \theta < 1$, $0 < p < 1$,

$$(23) \quad |f(z) - z| \leq \theta(1-p) + 2(\sqrt{1-p^2})^{\log \frac{1}{\theta}}$$

for $\frac{1}{2} \leq \theta < 1$, $0 < p < 1$,

we introduce the following simple inequality

$$(24) \quad \frac{4}{u^2 - 1} < \log \frac{(e^{\frac{2\pi}{\log u}} + 1)^2}{4e^{\log u}} \text{ for } 1 < u < \infty.$$

This can be proved as follows:

(i) $f_1(u) = \frac{2\pi}{\log u} - \frac{4}{u^2 - 1} - \log 4$ is a decreasing function of u for $1 < u < \infty$ moreover $f_1(e^4) = \frac{\pi}{2} - \frac{4}{e^4 - 1} - \log 4 > 0$, and so $f_1(u) > 0$

for $1 < u \leq e^1$, hence

$$\begin{aligned} \frac{4}{u^2-1} &< \frac{2\pi}{\log u} - \log 4 \\ &= \log \frac{e^{2\pi \log u}}{4} < \log \frac{(e^{\log u} + 1)^2}{4e^{\log u}} \text{ for } 1 < u \leq e^1. \end{aligned}$$

(ii) If we put

$$f_2(u) = \log \frac{(e^{\log u} + 1)^2}{4e^{\log u}} - \frac{4}{u^2-1},$$

then, for $e^1 < u < \infty$, we have

$$\begin{aligned} f_2'(u) &= \frac{8u}{(u^2-1)^2} - \frac{e^{\log u} - 1}{e^{\log u} + 1} \cdot \frac{2\pi}{(\log u)^2 u} \\ &< \frac{8u}{(u^2-1)^2} - \frac{\log u}{6} \cdot \frac{2\pi}{(\log u)^2 u} \\ &= \frac{8}{(\log u)^3 u} \left\{ \frac{u^2 (\log u)^3}{(u^2-1)^2} - \frac{\pi^2}{12} \right\} \\ &\leq \frac{8}{(\log u)^3 u} \left\{ \frac{u^2 (\log u)^3}{(u^2-1)^2} \right\}_{u=e^1} - \frac{\pi^2}{12} < 0. \end{aligned}$$

Hence $f_2(u)$ is a decreasing function of u for $e^1 < u < \infty$, moreover $\lim_{u \rightarrow \infty} f_2(u) = 0$, we have $f_2(u) > 0$ for $e^1 < u < \infty$. q.e.d.

Next, we shall prove that the following inequality holds.

$$\begin{aligned} \theta \sqrt{1-p^2} + 4p \sin^2 A &\leq \theta(1-p) \\ + \theta \sqrt{1-p^2} \sqrt{\log \frac{1}{1-\theta^2}}, \end{aligned}$$

where

$$0 < p < 1, \quad 0 \leq \theta < 1,$$

$$A = \text{Min} \left(\frac{\pi}{2}, \frac{1}{2\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta} \right).$$

As

$$\sqrt{1-p^2} + 4p \sin^2 A - (1-p) \leq 2p \sin A,$$

it is sufficient to show

$$(25) \quad 2p \sin A \leq \sqrt{1-p^2} \sqrt{\log \frac{1}{1-\theta^2}}.$$

(iii) When $1 \leq \frac{1}{2\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta}$, namely

$$1 > \theta \geq \frac{k-1}{k+1} \left(k = e^{\frac{2\pi}{\log \frac{1}{p}}} \right), \text{ if we put}$$

$u = \frac{1}{p}$ ($0 < p < 1$), from the inequality (24), we have

$$\begin{aligned} (26) \quad \frac{4p^2}{1-p^2} &< \log \frac{(k+1)^2}{4k} = \log \frac{1}{1 - \left(\frac{k-1}{k+1} \right)^2} \\ &\leq \log \frac{1}{1-\theta^2}. \end{aligned}$$

Therefore we have

$$\frac{2p \sin A}{\sqrt{1-p^2}} \leq \frac{2p}{\sqrt{1-p^2}} < \sqrt{\log \frac{1}{1-\theta^2}}.$$

(iv) When $0 \leq \frac{1}{2\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta} < 1$, namely

$$1 > \frac{k-1}{k+1} > \theta \geq 0, \text{ by (26)}$$

$$\frac{4}{\log \frac{(k+1)^2}{4k}} < \frac{1-p^2}{p^2}.$$

Considering as $\theta \neq 0$, we have

$$\begin{aligned} \frac{\pi^2(1-p^2)}{p^2 \left(\log \frac{1}{p} \right)^2} &> \frac{(\log k)^2}{\log \frac{(k+1)^2}{4k}} \\ &= \frac{\left(\log \frac{1+\theta}{1-\theta} \right)^2}{\log \frac{1}{1-\theta^2}} \bigg|_{\theta=\frac{k-1}{k+1}} > \frac{\left(\log \frac{1+\theta}{1-\theta} \right)^2}{\log \frac{1}{1-\theta^2}}, \end{aligned}$$

therefore we have

$$\begin{aligned} \sqrt{1-p^2} \sqrt{\log \frac{1}{1-\theta^2}} &> 2p \cdot \frac{1}{2\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta} \\ &= 2pA \geq 2p \sin A. \end{aligned}$$

For $\theta = 0$ it is evident that (25) is true. q.e.d.

Further we shall prove the following inequality holds.

$$(27) \quad \theta \sqrt{1-p^2} + 4p \sin^2 A < \theta(1-p)$$

$$+ 2(\sqrt{1-p^2})^{\log \frac{1}{\theta}},$$

where

$$\frac{1}{2} \leq \theta < 1, \quad 0 < p < 1,$$

$$A = \text{Min} \left(\frac{\pi}{2}, \frac{1}{2\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta} \right).$$

Putting $A = \frac{\pi}{2}$, we have

$$\theta p < \sqrt{1-p^2}^{\log \frac{1}{\theta}},$$

That is, it is sufficient to show

$$(28) \quad \frac{1}{2} \log \frac{1}{\theta} \log \frac{1}{1-p^2} < \log \frac{1}{\theta} + \log \frac{1}{p}.$$

Proof.

(v) When $0 < p \leq p^* = \frac{\sqrt{e^2-1}}{e}$, we have

$$\frac{1}{2} \log \frac{1}{1-p^2} \leq \frac{1}{2} \log \frac{1}{1-p^{*2}} = 1.$$

Hence we have (28).

(vi) When $1 > p > p^*$ and

$$\frac{1}{2\pi} \log \frac{1}{p} \cdot \log \frac{1+\theta}{1-\theta} \leq \frac{1}{2\pi},$$

namely $1 > \theta \geq \frac{k_1-1}{k_1+1}$ ($k_1 = e^{\frac{1}{\log \frac{1}{p}}}$),

in view of

$$1 > 2 \log \frac{1}{p} \log \frac{1}{1-p^2} - 6 \log \frac{1}{p},$$

we have

$$\begin{aligned} \frac{\log \frac{1}{p}}{\frac{1}{2} \log \frac{1}{1-p^2} - 1} &> \frac{4 \left(\log \frac{1}{p} \right)^2}{2 \log \frac{1}{p} + 1} > \frac{2}{e^{\log \frac{1}{p}} - 1} \\ &= \frac{2}{k_1 - 1} \geq \frac{1}{\theta} - 1 > \log \frac{1}{\theta}. \end{aligned}$$

That is, we have (28).

When $1 > p > p^*$, and

$$\frac{1}{2\pi} > \frac{1}{2\pi} \log \frac{1}{p} \log \frac{1+\theta}{1-\theta}, \text{ replacing}$$

$$\log \frac{1}{p} \log \frac{1+\theta}{1-\theta} \text{ with } \varepsilon \text{ and } \frac{\varepsilon}{\log \frac{1}{p}} \text{ with } \alpha$$

respectively, we have $\alpha = \log \frac{1+\theta}{1-\theta} > 1$,

$\varepsilon > \log \frac{1}{p}$, and it is sufficient to show

$$(29) \quad \frac{1}{2\pi} \varepsilon < \sqrt{1 - \log \frac{1}{\theta}}.$$

Taking logarithms of both side of (29),

$$2 \log 2\pi + 2 \log \frac{1}{\varepsilon} > \log \frac{1}{\theta} \log \frac{1}{1-p^2}.$$

On the other hand

$$\log \frac{1}{\theta} = \log \frac{e^x + 1}{e^x - 1} < \frac{2}{\alpha}$$

and so we have only to show

$$(30) \quad \log 2\pi + \log \frac{1}{\varepsilon} > \frac{1}{\alpha} \log \frac{1}{1-p^2}.$$

(vii) When $1 > p > p^*$ and $\frac{1}{2\pi} > \varepsilon > \log \frac{1}{p} > 0$,

$\varepsilon \log \frac{1}{\varepsilon}$ is an increasing function of ε .

Hence

$$(31) \quad \varepsilon \log \frac{1}{\varepsilon} > \log \frac{1}{p} \log \frac{1}{\log \frac{1}{p}}.$$

On the other hand

$$(32) \quad f_3(p) = \log \frac{1}{\log \frac{1}{p}} - \log \frac{1}{1-p^2} > 0,$$

because $f_3(p)$ is an increasing function of p for $1 > p > p^*$ and $f_3(p^*) > 0$.

By (31) and (32)

$$\log \frac{1}{\varepsilon} > \frac{1}{\varepsilon} \log \frac{1}{p} \log \frac{1}{1-p^2} = \frac{1}{\alpha} \log \frac{1}{1-p^2}.$$

This proves that (30) is true.

(viii) When $1 > p > p^*$ and $1 > \varepsilon > \frac{1}{2\pi}$, we have

$$(33) \quad \log \frac{1}{p^*} \log \frac{1}{1-p^{*2}} > \log \frac{1}{p} \log \frac{1}{1-p^2}.$$

$$\text{By (33) and } \alpha = \frac{\varepsilon}{\log \frac{1}{p}} > \frac{1}{2\pi \log \frac{1}{p}},$$

we have

$$\begin{aligned} \log 2\pi &> 2\pi \log \frac{1}{p^*} \log \frac{1}{1-p^{*2}} \\ &> 2\pi \log \frac{1}{p} \log \frac{1}{1-p^2}. \end{aligned}$$

From this we can get (30). This completes the proof.

At the end I wish to express my hearty thanks to Prof. A. Kobori of the Kyoto University, Prof. Y. Mori of the Kyoto Gakugei University and my friend Mr. Y. Kusunoki for their kind leadings.

References

- (1) Hurwitz-Courant, Funktionentheorie. 3. Aufl. Berlin. (1929)
- (2) Ostrowski, A. Über konforme Abbildungen annähernd kreisförmiger Gebiete. Jahresber. deutsch. Math.-Verein (1930), 78-81
- (3) Müller, M. Zur konforme Abbildung annähernd kreisförmiger Gebiete. Math. Zeitschr. (1938) 628-636
- (4) Nehari, Z. Conformal mapping. (1952)