

An Example of an ASL on a Distributive Lattice whose Straightening Relations Do Not Give a Presentation

Mitsuhiro MIYAZAKI

分配束上の ASL で、直列化関係式がその表示を与えないものの例

宮 崎 充 弘

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Abstract : DeConcini-Eisenbud-Procesi [1] proved that if $A = \bigoplus_{n \geq 0} A_n$ is a graded ASL on a partially ordered set H over $R = A_0$, then the straightening relations give a presentation of A , i. e., all the relations of the monomials on H are generated by the straightening relations. On the other hand, Trung [2] constructed counterexamples of the problem above (i. e., ASL's whose straightening relations do not give presentations) for general ASL's. But the partially ordered set of the example of Trung is not so good. So in this note, we construct an example of an ASL on a distributive lattice whose straightening relations do not give a presentation.

Key Words : ASL, Hodge Algebra, straightening relation

抄 録 : DeConcini-Eisenbud-Procesi [1] は $A = \bigoplus_{n \geq 0} A_n$ が順序集合 H 上の次数付き ASL であれば、その直列化関係式が A の表示を与える、すなわち、 H の単項式に関する関係式は、すべて直列化関係式で生成されることを示した。一方 Trung [2] は一般の ASL に関する上記の問題の反例（すなわち、直列化関係式がその表示を与えないような ASL の例）を構成した。しかし、Trung の例の順序集合は良い性質をあまり持っていない。そこで本論文では、分配束上の ASL で、直列化関係式がその表示を与えないものの例を構成する。

索引語 : ASL, Hodge 代数, 直列化関係式

Let R be a commutative ring and H be a finite partially ordered set (poset for short). An algebra with straightening law (ASL for short, ordinal Hodge algebra in the terminology of DeConcini-Eisenbud-Procesi [1]) on H over R is an R algebra A with an injection $\phi : H \longrightarrow A$ such that

(ASL-1) The family $\{\phi(\alpha_1) \cdots \phi(\alpha_n) \mid \alpha_i \in H (1 \leq i \leq n), \alpha_1 \leq \cdots \leq \alpha_n\}$ of the elements of A form

a free basis of A as an R -module. (The elements of A of the form $\phi(\alpha_1)\cdots\phi(\alpha_n)$, where $\alpha_i \in H (1 \leq i \leq n)$ and $\alpha_1 \leq \cdots \leq \alpha_n$, are called the standard monomials.)

(ASL-2) If $\alpha, \beta \in H$ are incomparable ($\alpha \not\leq \beta$ for short) and

$$\begin{aligned} \phi(\alpha)\phi(\beta) &= \sum_i r_i \phi(\gamma_{i1}) \cdots \phi(\gamma_{i n(i)}) \\ (0 \neq r_i \in R, \gamma_{ij} \in H, \gamma_{i1} \leq \cdots \leq \gamma_{i n(i)}) \end{aligned} \quad (1)$$

is the unique expression for $\phi(\alpha)\phi(\beta)$ as a linear combination of distinct standard monomials, then

$$\gamma_{i1} < \alpha, \beta$$

for any i . (The relations (1) are called the straightening relations for A .)

If we put

$$B := R[X_\alpha | \alpha \in H] / (X_\alpha X_\beta - \sum_i r_i X_{\gamma_{i1}} \cdots X_{\gamma_{i n(i)}} | \alpha \not\leq \beta)$$

where X_α are indeterminates, there is a natural R -algebra epimorphism $B \longrightarrow A$ sending the residue class of X_α to $\phi(\alpha)$. DeConcini-Eisenbud-Procesi [1, Proposition 1.1] proved that if $A = \bigoplus_{n \geq 0} A_n$ is a graded R -algebra such that $A_0 = R$ and $\phi(\alpha)$ is a homogeneous element of positive degree for each $\alpha \in H$, then the natural map $B \longrightarrow A$ is an isomorphism, i. e., the straightening relations (1) give a presentation of A .

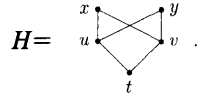
On the other hand, Trung [2] constructed ASL's whose straightening relations do not give presentations. So it is known that the natural map $B \longrightarrow A$ is not an isomorphism in general. But the poset in the examples of Trung [2] are rather bad and it is not even pure (i. e., there are maximal chains of length 2 and 3).

In this note we modify the example of Trung and construct an example of an ASL on a *distributive lattice* whose straightening relations do not give a presentation.

Example 1. Let R be an arbitrary commutative ring and X, Y, U, V, T are indeterminates. We put

$$\begin{aligned} A &= R[X, Y, U, V, T] / (XY - VY, UV - TUX, XYU) \\ &= R[x, y, u, v, t] \end{aligned}$$

and



Then

- A is an ASL on H over R whose straightening relations are

$$xy = vy, \quad uv = tx.$$

(ϕ is the inclusion map.)

- The natural map

$$B := R[X, Y, U, V, T]/(XY - VY, UV - TUX) \longrightarrow A$$

is not an isomorphism.

The proof of the second assertion is easy : if the natural map $B \longrightarrow A$ is an isomorphism, we have the following diagram

$$\begin{array}{ccc} B/(X - V, T - 1) & \simeq & A/(x - v, t - 1) \\ \downarrow & & \downarrow \\ R[X, Y, U] & & R[X, Y, U]/(XYU) \end{array}$$

and the natural map $R[X, Y, U] \longrightarrow R[X, Y, U]/(XYU)$ must be an isomorphism. Contradiction.

The proof of the first assertion is similar to the example of Trung [2, Example 2.2], but we give it for definiteness. First we prove the linear independence of the standard monomials. If the standard monomials are linearly dependent, we have a relation

$$\begin{aligned} & \sum_{m,n} (U(a_{mn}X^mU^n + Yb_{mn}Y^mU^n) + (c_{mn}X^mV^n + Yd_{mn}Y^mV^n)) \\ &= (XY - VY)f + (UV - UXT)g + XYUh \end{aligned}$$

in $R[X, Y, U, V, T]$ where $a_{mn}, b_{mn}, c_{mn}, d_{mn} \in R[T]$ and $f, g, h \in R[X, Y, U, V, T]$. If we write

$$f = f_1 + Uf_2, \text{ where } f_1 \in R[X, Y, V, T],$$

then we have

$$\sum_{m,n} (c_{mn}X^mV^n + Yd_{mn}Y^mV^n) = (XY - VY)f_1. \quad (2)$$

The right hand side of (2) contains a monomial which is a multiple of XY if $f_1 \neq 0$. Since the left hand side can not contain a monomial which is a multiple of XY , we see that $f_1 = 0$, i. e., $c_{mn} = d_{mn} = 0$ for any m and n . Similarly, if we put

$$g = g_1 + Yg_2, \text{ where } g_1 \in R[X, U, V, T],$$

we see that $g = Yg_2$ and $a_{mn} = 0$ for any m and n .

So we have

$$\sum_{m,n} b_{mn}Y^mU^n = (X - V)f_2 + (V - TX)g_2 + Xh. \quad (3)$$

Since the left hand side of (3) is contained in $R[Y, U, T]$ and the right hand side of (3) is contained in $(X, V)R[X, Y, U, V, T]$, we see that the both sides of (3) are 0, i. e., $b_{mn} = 0$ for any m and n .

Next we prove that A is spanned by the standard monomials as an R -module. First we note that

$$xy=vy, \quad uv=uxt, \quad xyu=yuv=0$$

in A . It is enough to show that every monomial of elements of H of the form xyM and uvM , where M is a monomial, is in the R -submodule of A spanned by the standard monomials.

xyM : If M is a multiple of u (considered in $R[X, Y, U, V, T]$), then $xyM=0$ since $xyu=0$. If M is not a multiple of u , we can rewrite xyM to a monomial on y, v , and t using the relations $x^n y = v^n y$ ($n=1, 2, \dots$).

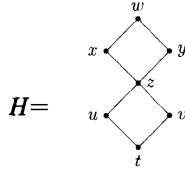
uvM : If M is a multiple of y , $uvM=0$ since $yuv=0$. If M is not a multiple of y , we can rewrite uvM to a monomial on x, u , and t using the relations $uv^n = u(xt)^n$ ($n=1, 2, \dots$).

Using Example 1, we can construct an ASL on a distributive lattice whose straightening relations do not give a presentation.

Example 2. Let R be an arbitrary commutative ring and X, Y, U, V, T, Z, W are indeterminates. We put

$$\begin{aligned} A &= R[X, Y, U, V, T, Z, W]/(XY-VY, UV-TUX, XYU) \\ &= R[x, y, u, v, t, z, w] \end{aligned}$$

and



Since A is a polynomial ring over the ASL in Example 1, we see that

- A is an ASL on H over R whose straightening relations are

$$xy=vy, \quad uv=tux.$$

(ϕ is the inclusion map.)

- The natural map

$$B := R[X, Y, U, V, T, Z, W]/(XY-VY, UV-TUX) \longrightarrow A$$

is not an isomorphism.

References

- [1] DeConcini, C., Eisenbud, D. and Procesi, C. : Hodge Algebras. Astérisque 91 (1982)
- [2] Trung, N. V. : On the presentation of Hodge algebras and the existence of Hodge algebra structures. Comm. Algebra 19 no. 4, 1183-1195 (1991)