Analysis of Pulse Modulated Control Systems (III)

Stability of Systems with Pulse Frequency Modulation and Systems with Combined Pulse Frequency and Pulse Width Modulation

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パルス変調制御システムの解析（Ⅲ）

パルス周波数変調システムおよびパルス周波数パルス幅複合変調システムの安定性

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Abstract: Sufficient conditions for finite pulse stability of interconnected systems with combined pulse frequency and pulse width modulation are developed in this paper using a direct method. The stability criteria established provide upper bounds on the number of pulses emitted by each modulator. The results are also applicable to those systems which contain a finite number of pulse frequency modulators and a finite number of combined pulse frequency and pulse width modulators.

Key Words: finite pulse stability, bound, combined pulse modulation

1. Introduction

Combined pulse frequency and pulse width modulation (PFPWM) is a process by which information of an input signal to the modulator is converted into emission time, polarity and width of the modulator output rectangular pulses of constant height. The pulse emission time is first decided by pulse frequency modulation (PFM) law then pulse width by pulse width modulation (PWM) law. Therefore, combined pulse frequency and pulse width modulation is a generalization of pulse frequency modulation. Kuntsevich and Chekhovoi (1967, 1971) were the first to propose this double modulation scheme for feedback systems combining static PFM (or PFM of the first kind) and PWM. They investigated stability of the systems using the second method of Lyapunov. In addition to static PFM-PWM, Varadarajan and Pai (1971) proposed double modulation schemes combining two types of dynamic PFM (or PFM of the second kind) and PWM, namely, integral PFM-PWM (IPFM-PWM) and neural PFM-PWM (NPFM-PWM). They established a nonlinear discrete equivalence for the systems. Datta (1971) investigated stability of IPFM-PWM feedback systems using the second method of Lyapunov. He also studied time response and periodic motion of the system. Oi and Kuwahara (1974, 1976) proposed a double modulation scheme for feedback systems combining generalized form of neural PFM and PWM, which is considered in this paper. They studied optimal control problems and periodic behavior of the systems, which were reported previously in this series of papers. Later, they investigated stability of the systems.

In this paper, a criteria for stability of interconnected systems containing arbitrary (finite) number of PFPWM modulators and a linear dynamic subsystem is presented. While stability of single modulator single feedback PFPWM systems have been studied by a few investigators, very few have been reported for multi-modulator multi-feedback PFPWM systems. Gülçür and Meyer (1973) investigated and obtained criteria for stability of multi-modulator multi-feedback PFM systems using a direct approach, namely, the direct application of basic functional properties of the system equations. This approach is employed in this paper.

2. Combined Pulse Frequency and Pulse Width Modulation

Fig. 1 shows a functional block diagram of the PFPWM modulator under consideration. PFM block is consisted of a resettable linear element and a threshold device. The k-th pulse emission time t_k is decided as follows. Both the absolute value of the modulator input $e(t)$ and the internal positive constant bias E (which is smaller than threshold a) are applied to the linear element which has impulse response $g_0(t) = \exp(-t/\tau_0)/\tau_0$, where τ_0 is time constant. The k-th pulse emission time t_k is the first time when the output $v(t)$ of the linear element is equal to or greater than threshold a after the time $t_{k-1} + T_m$, where T_m denotes minimum pulse interval. At the instant $t = t_k$, the linear element is reset and, at the same time, PWM block is activated to emit a rectangular pulse of constant height h which has the same sign as the input at t_k. Thus, t_k is defined by

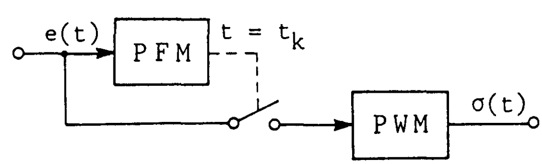

Fig.1 Block diagram of a PFPWM modulator

$$\left.\begin{aligned} t_k &= \min[t: t > t_{k-1} + T_m,\ v(t) \geq a] \\ v(t) &= \frac{1}{\tau_0}\int_{t_{k-1}}^{t} \exp\left(\frac{\tau - t}{\tau_0}\right)(\,|e(\tau)| + E\,)d\tau \end{aligned}\right\} \quad (1)$$

The pulse width w_k of the k-th pulse is defined to be proportional to the magnitude of the input signal at t_k, that is,

$$w_k = \min[k_0 \mid e(t_k) \mid, w_M] \tag{2}$$

where k_0 is a positive constant and w_M denotes maximum pulse width which must be not greater than T_m, i.e. $w_M \leq T_m$, to avoid overlapping of output pulses. Thus, the output of the modulator is given by

$$\sigma(t) = \sum_{k=1}^{N(t)} \mathrm{sgn}[e(t_k)]\, h\, [H(t-t_k) - H(t-t_k-w_k)] \tag{3}$$

where H(t) is the unit step function and N(t) denotes the total number of pulses emitted by the modulator prior to time t, that is, $t_{N(t)} \leq t < t_{N(t)+1}$. N(t) is a non-decreasing function of t and it takes nonnegative integer value, increasing stepwise at each pulse emission time. The combined pulse modulation defined by equations (1)−(3) is referred to extended neural PFM-PWM (ENPFM-PWM).

If the PWM block in Fig.1 is replaced by a pulse device which generates pulses of equal shape, a PFM modulator will be obtained. This type of PFM is referred to extended neural PFM (ENPFM) or neural PFM with bias. In this case, equation (2) should be ignored and equation (3) should be replaced by

$$\sigma(t) = \sum_{k=1}^{N(t)} \mathrm{sgn}[e(t_k)] \frac{1}{w} [H(t-t_k) - H(t-t_k-w)] \tag{3$'$}$$

in the case that the modulator output pulses are unit area rectangular pulses of pulse width w, or equation (3) should be replaced by

$$\sigma(t) = \sum_{k=1}^{N(t)} \mathrm{sgn}[e(t_k)]\, \delta(t-t_k) \tag{3$''$}$$

in the case that the modulator output pulses are unit impulses, where $\delta(t)$ is the unit impulse function.

3. Stability Criteria

3.1 Lyapunov's Second Method

The system under consideration is a single feedback system consisted of an ENPFM-PWM modulator and a linear dynamical subsystem. The modulator output $\sigma(t)$ is applied to the subsystem. The modulator input e(t) is the difference between the system input r(t) and the system output z(t). The system is described as follows:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}\sigma(t) \tag{4}$$

$$z(t) = \mathbf{d}^T \mathbf{x}(t) \tag{5}$$

$$\sigma(t) = \sum_i \mathrm{sgn}[e(t_i)]\, h\, [H(t-t_i) - H(t-t_i-w_i)] \tag{6}$$

$$\left.\begin{aligned} t_i &= \min[t: t > t_{i-1} + T_m,\ v(t) \geq a] \\ v(t) &= \frac{1}{\tau_0}\int_{t_{i-1}}^{t} \exp\left(\frac{\tau - t}{\tau_0}\right)(\mid e(\tau) \mid + E)\, d\tau \end{aligned}\right\} \tag{7}$$

$$w_i = \min[k \mid e(t_i) \mid, w_M] \tag{8}$$

where $\mathbf{x}(t)$ is the n-dimensional state vector of the subsystem, A is $n \times n$ constant coefficient matrix, $\mathbf{b}$ and $\mathbf{d}$ are n-dimensional constant vectors.

Stability is defined in terms of upper bounds on the number of pulses emitted by a modulator. The number of pulses emitted by a modulator is considered to be a measure of the energy spent by that modulator during the operation of the system.

Definition 1: A PFPWM system is called globally finite pulse stable (GFPS) if under any set of initial conditions the number of pulses emitted by the modulator remains finite as $t \to \infty$.

Let the combined state vector of the total system be denoted by

$$\mathbf{y}(t) = \begin{vmatrix} \mathbf{x}(t) \\ v(t) \end{vmatrix} \tag{9}$$

Theorem 1: If there exists a positive scalar function $V(\mathbf{y})$ and a constant $\varepsilon > 0$, such that for all $i = 1, 2, \cdots, N$,

$$V[\mathbf{y}(t_i + w_i)] - V[\mathbf{y}(t_{i+1} + w_{i+1})] > \varepsilon \tag{10}$$

then the PFPWM system is GFPS.

(Proof): From inequality (10), the following inequality is obtained

$$V[\mathbf{y}(t_N + w_N)] < V[\mathbf{y}(t_1 + w_1)] - N\varepsilon$$

In the above inequality N represents the total number of pulses emitted by the modulator as $t \to \infty$. Assume that $V[\mathbf{y}(t_1 + w_1)]$ is finite. Then N must also be finite, otherwise the above inequality yields $V[\mathbf{y}(t_N + w_N)] < -\infty$, which is the contradiction, since $V(\cdot)$ is a positive function. (QED)

Example 1 Consider the simple ENPFM-PWM system whose subsystem is an integrator of gain C, in the case the system input $r = 0$. Let $v(\mathbf{y}) = z^2$.

$$\begin{aligned} V&[z(t_i + w_i)] - V[z(t_{i+1} + w_{i+1})] \\ &= [z(t_i + w_i)]^2 - [z(t_i + w_i) - Chw_{i+1}\,\mathrm{sgn}[z(t_{i+1})]]^2 \\ &= Chw_{i+1}[2|z(t_i + w_i)| - Chw_{i+1}] > \varepsilon > 0 \\ &2|z(t_i + w_i)| > Chw_{i+1} \end{aligned}$$

Considering $|z(t_i + w_i)| > a - E = \Delta$ and $w_{i+1} \leq w_M$, C must be selected such that $2(a - E) > Chw_M$.

Then the system will be GFPS and will be stable in the sense of Lyapunov.

3.2 Stability of Interconnected System with Combined Pulse Frequency and Pulse Width Modulation

The system under consideration is illustrated in Fig.2. The modulator block is consisted of m ENPFM-PWM modulators (m = 1, 2, ---). The linear dynamical subsystem is consisted of m × m linear dynamical elements whose impulse responses are $g_{ij}(t)$ (i, j = 1, 2, ---, m) which are the elements of impulse response matrix G(t). The i-th modulator output $\sigma_i(t)$ (i = 1, 2, ---, m), which is the component of vector $\sigma(t)$, is applied to the linear dynamical elements. The i-th modulator input $e_i(t)$ is the difference between the external input $r_i(t)$ and the system output $x_i(t)$ (i = 1, 2, ---, m). $x_{0i}(t)$, which is the component of vector $\mathbf{x}_0(t)$, is the initial condition, response of the linear subsystem which could also include disturbances. $\mathbf{r}(t)$, $\mathbf{e}(t)$, $\sigma(t)$, $\mathbf{x}(t)$ and $\mathbf{x}_0(t)$ are m-dimensional vectors and G(t) is a m × m matrix.

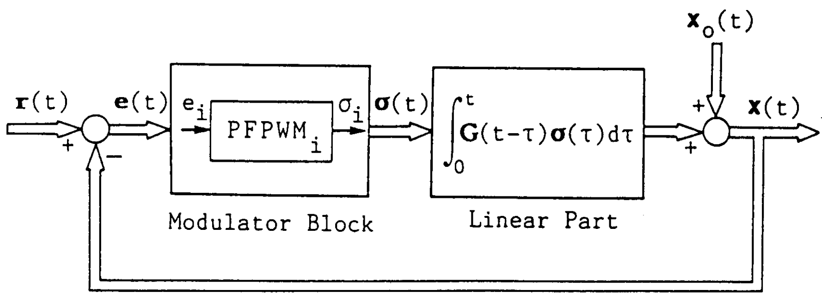

Fig.2 An Interconnected System with Combined Pulse Frequency and Pulse Width Modulation

Let $t_{i,k}$ be the instant at which the i-th modulator $PFPWM_i$ emits its k-th pulse and let $N_i(t)$ denote the total number of pulses emitted by the modulator prior to time t, that is, $t_{i,N_i(t)} \leq t < t_{i,N_i(t)+1}$. The system is defined as follows:
the system output $x_i(t)$ (i = 1, 2, ---, m):

$$x_i(t) = x_{0i}(t) + \sum_{j=1}^{m} \int_0^t g_{ij}(t-\tau)\sigma_j(\tau)d\tau \tag{11}$$

the i-th modulator $PFPWM_i$ (i = 1, 2, ---, m):

$$\left.\begin{aligned} t_{i,k} &= \min[t: t_{i,k-1} + T_{m_i} < t,\ v_i(t) \geq a_i] \\ v_i(t) &= \frac{1}{\tau_i}\int_{t_{i,k-1}}^{t} \exp\left(\frac{\tau - t}{\tau_i}\right)(\,|e_i(\tau)| + E_i)d\tau \end{aligned}\right\} \tag{12}$$

$$t_{i,k} = \min[k_i\ |e_i(t_{i,k})|\ ,\ w_{M_i}] \tag{13}$$

$$\sigma_i(t) = \sum_{k=1}^{N_i(t)} \mathrm{sgn}[e_i(t_{i,k})]\, h_i\, [H(t-t_{i,k}) - H(t-t_{i,k}-w_{i,k})] \tag{14}$$

$$e_i(t) = r_i(t) - x_i(t) \tag{15}$$

where notations of the i-th modulator are as follows:
$\sigma_i(t)$: the output pulses of the modulator, $e_i(t)$: the input to the modulator, H(t): unit

step function, T_{m_i} : minimum pulse interval, a_i : threshold, τ_i : time constant, E_i : positive bias ($< a_i$), $w_{i,k}$: pulse width of the k-th pulse, k_i : positive constant, w_{M_i} : maximum pulse width ($\leq T_{m_i}$), h_i : pulse height. The dead zone Δ_i of the modulator is $\Delta_i = a_i - E_i$.

Definition 2: An interconnected PFPWM system is called globally finite pulse stable (GFPS) if under any set of initial conditions the number of pulses emitted by each modulator remains finite as $t \to \infty$.

After all the modulators have ceased firing, the linear subsystem will remain without input and its motion can be studied independently by standard methods.

Here, the following assumptions are made:

Assumption 1: $x_{oi}(t), r_i(t) \in L_1[0, \infty)$ for $i = 1, 2, ---, m$.

Assumption 2: $g_{ij}(t) \in L_1[0, \infty)$ for $i, j = 1, 2, ---, m$.

Using equation (14), equation (11) yields

$$\begin{aligned} x_i(t) &= x_{oi}(t) + \sum_{j=1}^{m} \sum_{k=1}^{N_j(t)} \Big\{ \int_{t_{j,k}}^{t} g_{ij}(t-\tau)\,\mathrm{sgn}[e_j(t_{j,k})] \\ &\qquad \cdot h_j\,[H(\tau - t_{j,k}) - H(\tau - t_{j,k} - w_{j,k})] \Big\} d\tau \\ &= x_{oi}(t) + \sum_{j=1}^{m} \sum_{k=1}^{N_j(t)} \mathrm{sgn}[e_j(t_{j,k})]\, f_{ij}(t, t_{j,k}), \end{aligned} \tag{16}$$

where

$$\begin{aligned} f_{ij}(t, t_{j,k}) &= h_j \int_{t_{j,k}}^{t} g_{ij}(t-\tau)[H(\tau - t_{j,k}) - H(\tau - t_{j,k} - w_{j,k})]d\tau \\ &= \begin{cases} 0 & (t \leq t_{j,k}) \\ h_j \displaystyle\int_0^{t-t_{j,k}} g_{ij}(t - t_{j,k} - p)dp & (t_{j,k} < t < t_{j,k} + w_{j,k}) \\ h_j \displaystyle\int_0^{w_{j,k}} g_{ij}(t - t_{j,k} - p)dp & (t_{j,k} + w_{j,k} \leq t). \end{cases} \end{aligned} \tag{17}$$

From equation (12),

$$a_i \leq v_i(t_{i,k}) = \frac{1}{\tau_i} \int_{t_{i,k-1}}^{t_{i,k}} \exp\left(\frac{\tau - t_{i,k}}{\tau_i}\right)(\,|\,r_i(\tau) - x_i(\tau)\,| + E_i)d\tau, \tag{18}$$

thus,

$$\Delta_i = a_i - E_i \leq \frac{1}{\tau_i} \int_{t_{i,k-1}}^{t_{i,k}} [\,|\,r_i(\tau)\,| + |\,x_i(\tau)\,|\,]d\tau \qquad (k = 1, 2, ---, N_i(t)). \tag{19}$$

Using equality (16), the above inequality yields

$$\Delta_i \leq \frac{1}{\tau_i} \int_{t_{i,k-1}}^{t_{i,k}} [\,|\,r_i(\tau)\,| + |\,x_{oi}(\tau)\,| + \sum_{j=1}^{m} \sum_{n=1}^{N_j(\tau)} |\,f_{ij}(\tau, t_{j,n})\,|\,]d\tau \qquad (k = 1, 2, ---, N_i(t)). \tag{20}$$

Summing these inequalities for all intervals $[t_{i,0}, t_{i,1}]$, $[t_{i,1}, t_{i,2}]$, ---, $[t_{i,N_i(t)-1}, t_{i,N_i(t)}]$, assuming $t_{i,0} = 0$, yields

$$N_i(t)\Delta_i \leq \frac{1}{\tau_i}\int_0^{t_{i,N_i(t)}}[\,|r_i(\tau)| + |x_{0_i}(\tau)| + \sum_{j=1}^{m}\sum_{n=1}^{N_j(\tau)}|f_{ij}(\tau, t_{j,n})|\,]d\tau \quad (21)$$

$$(i = 1, 2, \text{---}, m).$$

Recognizing that $t_{i,N_i(t)} \leq t$, the upper limit of the integral in inequality (21) can be extended from $t_{i,N_i(t)}$ to t. Considering the following relation

$$\int_0^t \sum_{j=1}^{m}\sum_{n=1}^{N_j(\tau)}|f_{ij}(\tau, t_{j,n})|\,d\tau \leq \sum_{j=1}^{m}\sum_{n=1}^{N_j(\tau)}\int_0^t |f_{ij}(\tau, t_{j,n})|\,]d\tau$$

$$\leq \sum_{j=1}^{m}\sum_{n=1}^{N_j(\tau)}\int_{t_{j,n}}^t [h_j\int_0^{w_{j,n}}|g_{ij}(\tau - t_{j,n} - p)|\,dp\,]d\tau$$

$$\leq \sum_{j=1}^{m} N_j(t)\,h_j\int_0^t [\int_0^{w_{M_j}}|g_{ij}(\tau - p)|\,dp\,]d\tau, \quad (22)$$

the inequality (21) yields

$$N_i(t) \leq z_i(t) + \sum_{j=1}^{m} N_j(t)\,h_{ij}(t), \qquad (i = 1, 2, \text{---}, m) \quad (23)$$

or in vector form:

$$[\mathbf{I} - \mathbf{H}(t)]\,\mathbf{n}(t) \leq \mathbf{z}(t) \quad (24)$$

where inequality stands for the component-wise inequality and where $\mathbf{n}(t)$ is a m-dimensional vector defined by

$$\mathbf{n}(t) = [N_1(t), N_2(t), \text{---}, N_m(t)]^T. \quad (25)$$

$z(t)$ is a m-dimensional vector whose i-th element is defined by

$$z_i(t) = \frac{1}{\tau_i\ \Delta_i}\int_0^t [\,|r_i(\tau)| + |x_{0_i}(\tau)|\,]d\tau \quad (26)$$

and $\mathbf{H}(t)$ is a m × m matrix whose element in the i-th row and j-th column is

$$h_{ij}(t) = \frac{h_j}{\tau_i\ \Delta_i}\int_0^t [\int_0^{w_{M_j}}|g_{ij}(\tau - p)|\,dp\,]d\tau. \quad (27)$$

When the matrix $[\mathbf{I} - \mathbf{H}(t)]^{-1}$ exists and is element-wise positive, inequality (24) can be transformed into

$$\mathbf{n}(t) \leq \text{integer}\{[\mathbf{I} - \mathbf{H}(t)]^{-1}\mathbf{z}(t)\} \quad (28)$$

where the notation integer{ } stands for the integer part of the corresponding vector. Inequality (28) determines the upper bounds on the number of pulse emissions as $t \to \infty$.

These results, together with Definition 2, may be stated in terms of the following theorem.

Theorem 2: If Assumptions 1 and 2 are satisfied and if there exists $\lim_{t\to\infty}[\mathbf{I} - \mathbf{H}(t)]^{-1} = [\mathbf{I} - \mathbf{H}_\infty]^{-1}$ and is element-wise positive, then the system of Fig.2 is GFPS.

Lemma 1: If Assumptions 1 and 2 are satisfied and if the system of Fig.2 is GFPS, then $x_i(t) \in L_1\,[0, \infty)$ for i = 1, 2, ---, m.

(Proof): Consider the system output $x_i(t)$ (i = 1, 2, ---, m)

$$x_i(t) = x_{oi}(t) + \sum_{j=1}^{m} \sum_{n=1}^{N_j(t)} \mathrm{sgn}\,[e_j\,(t_{j,n})]\, f_{ij}(t, t_{j,n}). \tag{29}$$

For $t \geq t_{i,N_i(t)}$ (i = 1, 2, ---, m),

$$\int_0^t |\,x_i(\tau)\,|\,d\tau \leq \int_0^t |\,x_{oi}(\tau)\,|\,d\tau + \int_0^t \sum_{j=1}^{m} \sum_{n=1}^{N_j(\tau)} |\,f_{ij}(\tau, t_{j,n})\,|\,d\tau$$

$$\leq \int_0^t |\,x_{oi}(\tau)\,|\,d\tau + \sum_{j=1}^{m} N_j\,(t)\,h_j \int_0^t [\int_0^{w_{M_j}} |\,g_{ij}(\tau - p)\,|\,dp\,]\,d\tau. \tag{30}$$

From the hypothesis of Lemma 1,

$$\lim_{t \to \infty} \int_0^t |\,x_i(\tau)\,|\,d\tau \leq \lim_{t \to \infty} \int_0^t |\,x_{oi}(\tau)\,|\,d\tau$$

$$+ \sum_{j=1}^{m} N_j\,(\infty)\,h_j \int_0^{w_{M_j}} [\lim_{t \to \infty} \int_0^t |\,g_{ij}(\tau - p)\,|\,d\tau\,]\,dp < \infty. \tag{31}$$

that is, $x_i(t) \in L_1\,[0, \infty)$.

4. Applications

The Theorem 2 can be applied to the following cases.

4.1 Application to single modulator single feedback systems

(a) A feedback system with an ENPFM-PWM modulator

In this case, the theorem imposes the following stability condition

$$\Delta > \lim_{t \to \infty} \frac{h}{\tau_1} \int_0^t [\int_0^{w_M} |\,g(\tau - p)\,|\,dp\,]\,d\tau. \tag{32}$$

(b) A feedback system with an ordinary NPFM-PWM modulator

In this case, E = 0, that is $\Delta = a$. Replacing Δ by α in inequality (32), the stability condition is obtained.

(c) A feedback system with a neural PFM modulator of rectangular pulse output type

In this case, pulse width is kept constant, that is, $w_k = w_M = w =$ const. and $h = 1/w$. The stability condition is given by

$$\Delta > \lim_{t \to \infty} \frac{1}{\tau_1\, w} \int_0^t [\int_0^{w} |\,g(\tau - p)\,|\,dp\,]\,d\tau. \tag{33}$$

(d) A feedback system with a neural PFM modulator of impulse output type

Making $w \to 0$ in inequality (33), the stability condition is obtained as follows:

$$a > \lim_{t \to \infty} \frac{1}{\tau_1} \int_0^t | g(\tau) | d\tau . \qquad (34)$$

Inequality (34) was obtained by Gülçür and Meyer (1973) and by Skoog and Blankenship (1970).

4.2 Application to multi-modulator multi-feedback systems

(a) An interconnected system with ENPFM-PWM modulators

The theorem can be applied directely without any modifications. An example of a simple multi-feedback system with two modulators is presented in this section.

(b) An interconnected system with PFM modulators and PFPWM modulators

Suppose q modulators ($q \leq m$) are PFM modulators (extended neural PFM modulator of rectangular pulse output type) and $m - q$ modulators are PFPWM (ENPFM-PWM) modulators. In this case, to the corresponding PFM modulators, pulse width is kept constant, that is,

$$w_{i,k} = w_{M_i} = w_i = \text{const. and } h_i = 1/w_i \qquad (i = 1, \text{---}, q).$$

The theorem will be effective for this case, if the equation (27) is replaced by

$$\left.\begin{aligned} h_{ij}(t) &= \frac{1}{\tau_i \Delta_i w_i} \int_0^t [\int_0^{w_j} | g_{ij}(\tau - p) | dp] d\tau , \quad (j = 1, \text{---}, q) \\ h_{ij}(t) &= \frac{h_j}{\tau_i \Delta_i} \int_0^t [\int_0^{w_{M_j}} | g_{ij}(\tau - p) | dp] d\tau , \quad (j = q+1, \text{---}, m) \end{aligned}\right\} \qquad (35)$$

In a special case that all the modulators are neural PFM modulators of impulse output type, the equation (27) should be replaced by

$$h_{ij}(t) = \frac{1}{\tau_i a_i} \int_0^t | g_{ij}(\tau) | d\tau . \qquad (36)$$

In this special case, the theorem coincides with the results by Gülçür-Meyer's theorem applied to neural PFM interconnected systems.

Example Consider a PFPWM interconnected system shown in Fig.3. In this case, the linear subsystem has the impulse response matrix

$$\mathbf{G}(t) = \begin{vmatrix} A_1 \{\exp(-\alpha_1 t) + \exp(-\alpha_2 t)\} & A_2 \exp(-\alpha_6 t) \\ A_1 \exp(-\alpha_5 t) & A_2 \{\exp(-\alpha_3 t) + \exp(-\alpha_4 t)\} \end{vmatrix} . \qquad (37)$$

From equation (22), the matrix $\mathbf{H}_\infty$ is computed as

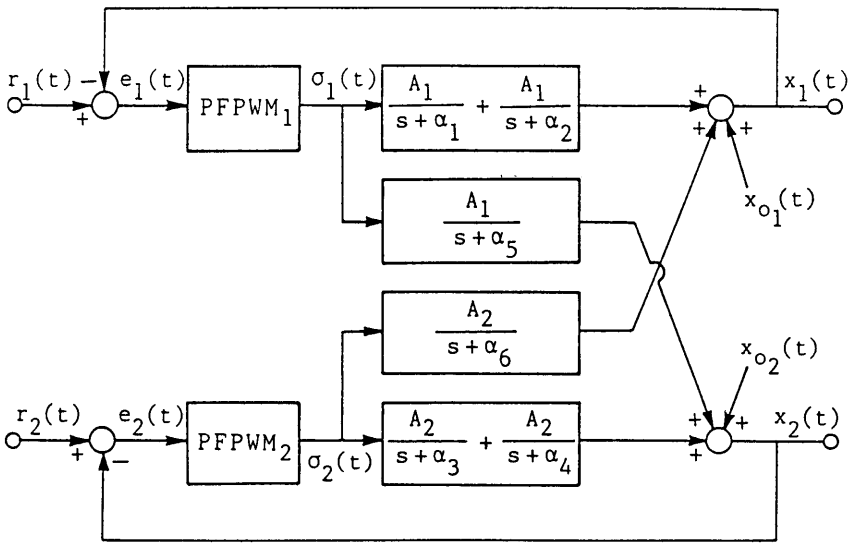

Fig.3 A PFPWM interconnected system

$$\mathbf{H}_\infty = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \tag{38}$$

where

$$\left.\begin{aligned}
h_{11} &= \frac{h_1 A_1}{\tau_1 \Delta_1}\left[\frac{\exp(\alpha_1 w_{M1})-1}{\alpha_1{}^2}+\frac{\exp(\alpha_2 w_{M1})-1}{\alpha_2{}^2}\right], \\
h_{21} &= \frac{h_1 A_1}{\tau_1 \Delta_1}\frac{\exp(\alpha_5 w_{M1})-1}{\alpha_5{}^2}, \\
h_{12} &= \frac{h_2 A_2}{\tau_2 \Delta_2}\frac{\exp(\alpha_6 w_{M2})-1}{\alpha_1{}^2}, \\
h_{22} &= \frac{h_2 A_2}{\tau_2 \Delta_2}\left[\frac{\exp(\alpha_3 w_{M2})-1}{\alpha_3{}^2}+\frac{\exp(\alpha_4 w_{M2})-1}{\alpha_4{}^2}\right].
\end{aligned}\right\} \tag{39}$$

Thus, if the following inequalities are satisfied, the matrix $[\mathbf{I}-\mathbf{H}_\infty]^{-1}$ exists and is positive, and the system is GFPS.

$$\left.\begin{aligned}
&(1-h_{11})(1-h_{22})-h_{12}h_{21} > 0\ , \\
&1-h_{11} > 0\ , \\
&1-h_{22} > 0\ .
\end{aligned}\right\} \tag{40}$$

Fig.4 shows an example of time response of the system shown in Fig.3, obtained by digital computer simulation, under the conditions:

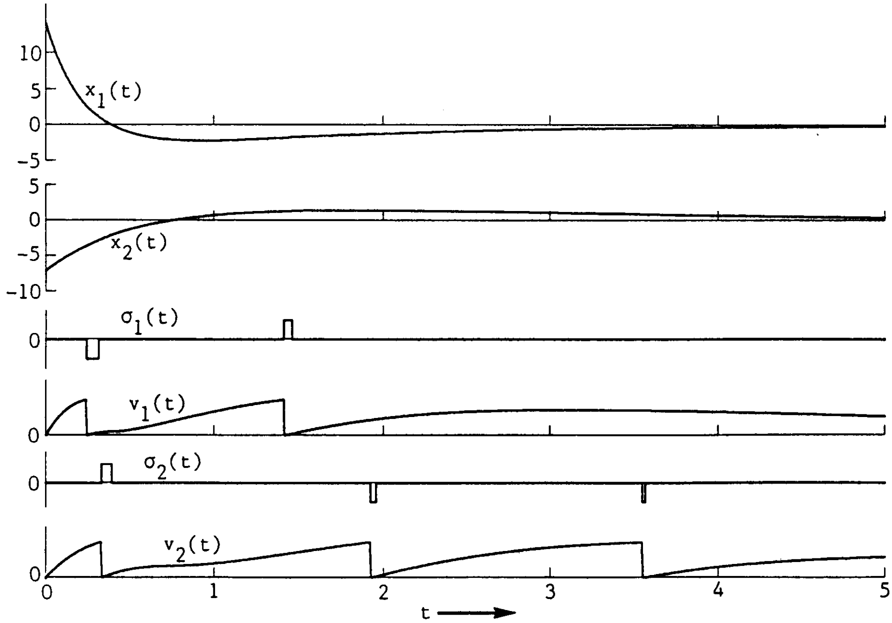

Fig.4 Time response of the system of Fig.3

inputs: $r_1(t) = r_2(t) = 0$

initial condition responses:

$$x_{o1}(t) = 8\exp(-t) + \exp(-2t) + 2\exp(-3t),$$
$$x_{o2}(t) = -8\exp(-3t) - 10\exp(-5t) + 6\exp(-t).$$

The system parameters are $a_1 = 1$, $E_1 = 0.2$, $1/\tau_1 = 0.6$, $w_{M1} = T_{m1} = 0.2168$, $k_1 = 0.027$, $h_1 = 3$, $a_2 = 1$, $E_2 = 0.3$, $1/\tau_2 = 0.7$, $w_{M2} = T_{m2} = 0.1834$, $k_2 = 0.023$, $h_2 = 2$, $A_1 = 0.2$, $A_2 = 0.3$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 1$.

In this case, the inequalities (40) are all satisfied and the system is GFPS. From inequality (28) for $t \to \infty$, the bounds on the number of pulse emissions are

$$\mathbf{n}_\infty \leq [8, 6]^T,$$

that is, $PFPWM_1$ will stop firing after at most 8 pulse emissions and $PFPWM_2$ after at most 6 pulse emission. The actual numbers of pulse firings for this system obtained by computer simulation are

$$\mathbf{n}_\infty = [4, 2]^T.$$

5. Conclusions

Sufficient conditions for finite pulse stability of interconnected systems containing a finite number of combined pulse frequency and pulse width modulators and a linear subsystem are developed using a direct method which is the direct application of basic functional properties of the system equations. The resusts are simple to apply and, at the same time, provide bounds on the number of pulses emitted by each modulator.

Though the results are for interconnected systems containing a finite number of PFPWM modulators, the results are also applicable to those systems which contain a finite number of PFM and PFPWM modulators.

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