

Note for the factorization of certain periodic entire functions

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ある種の周期整函数の分解について

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Introduction

An entire function $F(z)$ ($\neq \text{const.}$) is said to have a factorization with left-factor $f(z)$ and right-factor $g(z)$, when, under

$$F(z) = f(g(z)) = f \circ g(z), \quad (1)$$

f is meromorphic and g is entire (g may be meromorphic, when f is rational).

Now, $F(z)$ is called to be *prime* (*pseudo-prime*; *left-prime*; or *right-prime*, respectively), if, for every factorization such as (1), we can deduce the following assertion: f or g is linear (f is rational or g is a polynomial; f is linear whenever g is transcendental; or g is linear whenever f is transcendental, resp.). When factors are restricted to entire functions, the factorization is said to be *in entire sense* (F may be prime in entire sense, etc.).

Well, assume that a non-constant entire function $F(z)$ would have two factorizations:

$$F(z) = f_1 \circ f_2 \circ \dots \circ f_m(z) = g_1 \circ g_2 \circ \dots \circ g_n(z) \quad (2)$$

into non-linear entire factors. Here, if $m=n$ and if with suitable linear polynomials $T_j(z)$ ($j=1, \dots, n-1$) the following relations

$$f_1(z) = g_1 \circ T_1^{-1}(z), f_2(z) = T_1 \circ g_2 \circ T_2^{-1}(z), \dots, f_n(z) = T_{n-1} \circ g_n(z)$$

hold identically, then the two factorizations in (2) are called to be *equivalent* (in entire sense). If any two factorizations of $F(z)$ into non-linear, prime, entire factors are equivalent, we say that $F(z)$ is *uniquely factorizable*. Of course, every prime functions may be considered to be uniquely factorizable.

So far, several results have been obtained concerning the uniqueness of the factorization for certain classes of entire functions, along with the primeness etc. (cf. [11], [14], [16],

[17] and others). Note that there exist entire functions which need not be uniquely factorizable, and that the functions e^z and $\cos z$ have an infinite number of non-equivalent factorizations (see, for example, [14]).

In this note, continuing the preceding investigation ([16] and [17]), we shall prove some result (Theorem 1) which will be closely related to the uniqueness of the factorization. This result has been stated at the final section in [17], without proof.

Also, we shall note (and prove in detail) some fact (cf. Prop. in [17]) which shows that, for certain periodic entire function $f(z)$, the exponent of the convergence of the zeros of f is equal to the order of f (Theorem 2). Finally, we'll consider an example for which we can compute the exponent of the convergence of the zeros of the function.

For an entire function $f(z)$, we shall denote by $\rho(f)$ ($\underline{\rho}(f)$) the order (lower order, resp.) of f and by $\rho^*(f)$ the exponent of convergence of the zeros of $f(z)$. Here

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r},$$

where $M(r, f)$ is the maximum modulus of $f(z)$ for $|z|=r$ and $T(r, f)$ is the Nevanlinna characteristic function of $f(z)$ ($\underline{\rho}(f)$ is given by $\liminf$). Further $\rho^*(f)$ is known to be equal to the order of $N(r, 1/f)$ (see, for example, [6] p. 16-25).

1. The main result is stated as follows.

Theorem 1. Let

$$F(z) = (P(z) \cdot e^{Q(z)}) \circ (h(e^z)) = P(h(e^z)) \cdot \exp[Q(h(e^z))], \quad (3)$$

where $P(z)$, $Q(z)$ and $h(z)$ are entire functions which satisfy

$$\rho[P(h(e^z))] < \infty, \quad \underline{\rho}[Q(h(e^z))] < \infty \quad (4)$$

and that the function $P(h(z))$, $P(h(0)) \neq 0$, has at least one simple zero or two zeros with co-prime multiplicities.

Assume now that $F(z)$ is factorized such as

$$F(z) = f(g(z)) = f \circ g(z), \quad (5)$$

where f and g are two non-linear entire functions.

Then we have necessarily

$$f(z) = S(z) \cdot e^{A(z)} \quad (6)$$

and

$$g(z) = U(e^{z/k}) = U(z) \circ (e^{z/k}) \quad (7)$$

such that the following relations hold:

$$S(U(z)) = e^c \cdot P(h(z^k)) \quad (8)$$

and

$$A(U(z)) = Q(h(z^k)) - c, \quad (9)$$

where c is a constant, k is a natural number and that $A(z)$, $S(z)$ and $U(z)$ are some entire functions.

In particular, $g(z)$ under (5) must be periodic.

Remark 1. The conditions in (4) can be written as

$P(h(z)) \in \mathbf{H}$ and $Q(h(z)) \in \mathbf{H}$ (cf. Lemma 5 and also [19]). Of course, P , Q and h may be all transcendental.

Remark 2. The condition $P(h(0)) \neq 0$ cannot be dropped, This is seen by the following example:

$$F(z) = (z \cdot e^z) \circ (e^z) = (e^z) \circ (z + e^z).$$

In this case, $P(h(z)) \equiv z$.

2. Lemmas to prove Theorem 1.

Lemma 1. (Pólya [12]) Suppose $f(z)$, $g(z)$ and $h(z)$ are non-constant entire functions such that $f(z) = g(h(z))$. If $h(0) = 0$, then there exists a constant c with $0 < c < 1$ such that $M[cM(r/2, h), g] \leq M(r, f)$ ($r \geq r_0$).

Here the condition $h(0) = 0$ is not essential, which means that if we add the condition $r \geq r_0$ (for sufficiently large values of r), then the condition $h(0) = 0$ can be removed.

Note that the inequality

$$M(r, f) \leq M(M(r, h), g)$$

is valid naturally, by the maximum modulus principle.

Lemma 2. (Edrei-Fuchs [2]) Let $f(z)$ and $g(z)$ be two transcendental entire functions. If $\rho^*(f)$ is positive, then $\rho^*(f(g(z)))$ is infinite.

Lemma 3. (cf. [1] or [3]) Let $f(z) (\neq \text{const.})$ and $g(z)$ be two entire functions such that the order $\rho(f)$ is less than $1/2$. If $f(g(z))$ is periodic, then $g(z)$ must be so.

Lemma 4. (Kobayashi [7]) Let $f(z)$ be an entire function such that

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{r} = 0.$$

Assume that there exists an unbounded sequence $\{w_n\}$ such that all the roots of the equations

$$f(z) = w_n \quad (n=1, 2, \dots)$$

lie in a half plane ($\{\operatorname{Re} z \geq c\}$ for some real number c , say), then $f(z)$ is a polynomial of degree at most two.

Lemma 5. (Yang-Urabe [19]) Let $f(z)$ be an entire function of finite order and of positive lower order. Suppose that $S(z)$ is an entire function of order not greater than $1/2$ such that $S \in \mathbf{H}$, i.e.

$$(*) \quad \liminf_{r \rightarrow \infty} \frac{\log \log M(r, S)}{\log \log r} = \infty.$$

Then $\rho^*(S(f(z)))$ is infinite.

Lemma 6. (cf. [19]) Let $f(z)$ be an entire function of finite order. Then if $S \in \mathbf{H}$ (i.e. the left hand side of (*) in the above Lemma 5 is finite), the lower order $\underline{\rho}(S(f(z)))$ is finite.

Lemma 7. (cf. [19]) Let $f(z)$ be an entire function. If the lower order $\underline{\rho}(e^{f(z)})$ is finite, then $f(z)$ is a polynomial.

Lemma 8. Let

$$F(z) = H(z) + Q(z),$$

where $H(z)$ ($\neq \text{const.}$) is an entire function of finite lower order which is periodic with some non-zero period b ($H(z+b) \equiv H(z)$) and $Q(z)$ is a non-constant polynomial. Then $F(z)$ is left-prime. Further, if $Q(z)$ has no quadratic right factor, then $F(z)$ is prime.

About this Lemma 8, see Lemma 11' in [14]. The proof is given in [15]. This was a conjecture of Gross (cf. [4]).

3. Proof of Theorem 1.

Let

$$F(z) = f(g(z)) = f \circ g(z), \quad (10)$$

where f and g are two non-linear entire functions.

At first, we'll show that f and g , under (10), both must be transcendental. Now assume that f is a non-linear polynomial. If f has at least two distinct zeros, then by noting $\rho(g) = \infty$ in this case (since F is so), from Borel-Nevanlinna's theorem (cf. [8] p. 72), we know that $\rho^*(f(g(z))) = \infty$. However, this contradicts

$$\rho^*(f(g(z))) = \rho^*(P(h(e^z))) \leq \rho(P(h(e^z))) < \infty.$$

(Note that the above second inequality is indeed equality. About this assertion, see Theorem 2 below.) If f has only one zero point (z_0 , say), then f can be written as $f(z) = a(z-z_0)^m$ for some constant $a \neq 0$ and natural number $m \geq 2$. In this case, the multiplicities of every zeros of (10) and hence those for $P(h(e^z))$ must be some multiples of m . Since e^z has no multiple values, this contradicts the assumption concerning $P(h(z))$.

Next assume that g is a non-linear polynomial. Then the degree of g must be equal to 2. This is seen by noting the distribution of the zeros of $F(z)$ (cf. the results in [13], also). Hence

$$g(z) = a(z-z_0)^2 + b,$$

for some constants $a \neq 0$, b and z_0 . Putting $w = z - z_0$, we have

$$F(-w+z_0) \equiv F(w+z_0). \quad (11)$$

Now we can write

$$F(z) = L(e^z), \quad (12)$$

where

$$L(z) = P(h(z)) \cdot \exp[Q(h(z))].$$

Then from (11) and (12), we have the identity

$$L(e^{z_0} \cdot e^{-w}) = L(e^{z_0} \cdot e^w). \quad (13)$$

Since L is entire, the function $L(e^{z_0} \cdot e^w)$ is bounded in the left half plane, by nature. By (13) it is bounded also in the right half plane. Hence it becomes bounded in the whole plane so that by Liouville's theorem it must be constant. Then $L(z)$ and hence $F(z)$ (by (12)) is constant, which is not in reason, a contradiction.

Subsequently, we'll treat the following two cases, separately.

The case (i); $f(z)$ has only one simple zero.

The case (ii); $f(z)$ has at least two distinct zeros.

In each case (i) or (ii), as is shown above, f and g are both transcendental.

3.1. The case (i).

In this case, we may put

$$f(z) = z \cdot e^{A(z)}, \quad (14)$$

where $A(z)$ is a non-constant entire function.

From (3), (10) and (14), we have

$$g(z) = P(h(e^z)) \cdot e^{B(z)} \quad (15)$$

and

$$A(g(z)) = Q(h(e^z)) - B(z) \quad (16)$$

for some entire function $B(z)$. Here

$$B(z) \text{ is a polynomial,} \quad (17)$$

which follows from Lemma 7, in view of (4) and (15), if we can show that

$$\rho(g) < \infty. \quad (18)$$

By using Lemma 1, we can prove the assertion (18) as follows. From (14), we have for $r \geq r_0$

$$M(r, f) \geq e^{dr}$$

for some positive constant d . Since $\exp[Q(h(e^z))]$ is of infinite lower order, noting (4), we have

$$\begin{aligned} M(r, F) &\leq (1+o(1)) \cdot M(r, e^{Q(h(e^z))}) \leq (1+o(1)) \cdot M[M(r, Q(h(e^z))), e^z] \\ &= (1+o(1)) \cdot \exp[M(r, Q(h(e^z)))] \leq \exp[2M(r, Q(h(e^z)))]. \end{aligned}$$

While, by Lemma 1, we have

$$M(r, f(g(z))) \geq M(cM(r/2, g), f) \geq e^{cdM(r/2, g)}.$$

Here we also used the fact noted above. Since $F(z) = f(g(z))$, combining the above two inequalities, we obtain

$$M(r/2, g) \leq \frac{2}{cd} \cdot M(r, Q(h(e^z))) \quad (r \geq r_0).$$

Hence, noting the latter condition in (4), we see that (18) is valid. Thus (17) is valid, as is noted already.

If

$$B(z) \equiv \text{const.} = c \text{ (say),} \quad (19)$$

then from (15) and (16)

$$g(z) = e^c \cdot P(h(e^z)) \quad (20)$$

and

$$A(g(z)) = Q(h(e^z)) - c,$$

so that we have

$$A(e^c \cdot P(h(e^z))) = Q(h(e^z)) - c.$$

Hence

$$A(e^c \cdot P(h(z))) = Q(h(z)) - c. \quad (21)$$

Therefore, putting $S(z) = z$, $U(z) = e^c \cdot P(h(z))$ and $k=1$, from (14), (20) and (21) we have

$$f(z) = z \cdot e^{A(z)} = S(z) \cdot e^{A(z)}$$

and

$$g(z) = U(z) \circ (e^z)$$

such that

$$S(U(z)) = U(z) = e^c \cdot P(h(z))$$

and

$$A(U(z)) = Q(h(z)) - c.$$

These relations correspond to (6), (7), (8) and (9) resp.

Next, if

$$B(z) \neq \text{const. (polynomial)}, \quad (22)$$

applying Lemma 8, we know that the right hand side of (16) is left-prime. Since g is transcendental, this implies

$$A(z) = az + b; \text{ linear } (a \neq 0). \quad (23)$$

From (16) and (23), we have

$$g(z) = 1/a \cdot [Q(h(e^z)) - B(z) - b]. \quad (24)$$

Hence, by (15) and (24), the following identity must hold :

$$a \cdot P(h(e^z)) \cdot e^{B(z)} = Q(h(e^z)) - B(z) - b. \quad (25)$$

However, by comparing the growth of the both sides of (25) along the imaginary axis, we see easily that the identity (25) cannot hold, a contradiction. In fact, the right hand side (25) is unbounded and of polynomial growth along the imaginary axis, while the left hand side is either bounded (if $e^{B(z)}$ is so) or it has at least order-one-like-growth on the (upper or lower part of) imaginary axis. Thus we conclude that the case such as (22) does not occur.

3.2. The case (ii).

In this case, we can put

$$f(z) = S(z) \cdot e^{A(z)}, \quad (26)$$

where $S(z)$ is a non-linear polynomial with at least two distinct zeros or a transcendental entire function with an infinite number of zeros and $A(z)$ is an entire function. Note that when $S(z)$ is a polynomial $A(z)$ is non-constant, because f is transcendental. (As the function $S(z)$ in the latter case, we take the canonical product which is constructed from the zeros of $f(z)$ and hence $\rho^*(S) = \rho(S)$).

Now the relation (10) becomes

$$S(g(z)) \cdot e^{A(g(z))} = P(h(e^z)) \cdot e^{Q(h(e^z))}.$$

Hence

$$S(g(z)) = P(h(e^z)) \cdot e^{B(z)} \quad (27)$$

and

$$A(g(z)) = Q(h(e^z)) - B(z) \quad (28)$$

for some entire function $B(z)$.

In this case, by Borel-Nevanlinna's theorem (already used before), we know from (27) that

$$\rho(g) < \infty, \quad (29)$$

since $S(z)$ has at least two distinct zeros. Here, note that (4) is assumed and hence $\rho^*(S(g(z))) = \rho^*(P(h(e^z))) < \infty$.

Using (27) again, we'll show that

$$B(z) \text{ is a polynomial.} \quad (30)$$

If $S(z)$ is a polynomial, from (27) and (29), we see at once that the assertion (30) is valid, since $e^{B(z)}$ is of finite order. (cf. Lemma 7).

If $S(z)$ is a transcendental entire function, by applying Lemma 2 (noting that g is transcendental), we see at first

$$\rho(S) = 0. \quad (31)$$

Well, all the zeros of $F(z)$ are contained in a half-plane $\{\operatorname{Re} z \geq c\}$ (say). Hence those of $S(g(z))$ are so. Noting this fact and that g is transcendental, by Lemma 4, we have

$$\rho(g) \geq 1 (> 0). \quad (32)$$

Since $\rho^*(P(h(e^z))) = \rho^*(S(g(z))) < \infty$, noting (32) and applying Lemma 5, we have

$$S \in \underline{\mathbf{H}}, \quad (33)$$

so that by Lemma 6

$$\rho(S(g(z))) < \infty. \quad (34)$$

Hence, using (27), we obtain

$$\rho(e^{B(z)}) < \infty. \quad (35)$$

Applying Lemma 7 under (35), we see that (30) is valid, that is, $B(z)$ is a polynomial.

Assume that

$$B(z) \equiv \text{const.} = c(\text{say}). \quad (36)$$

Then, the relations (27) and (28) can be written as

$$S(g(z)) = e^c \cdot P(h(e^z)) \quad (37)$$

and

$$A(g(z)) = Q(h(e^z)) - c. \quad (38)$$

Here, note the fact (31). Then, applying Lemma 3 to (37), we see that g is periodic. Hence we may write

$$g(z) = r(e^{z/k}) + t(e^{-z/k}), \quad (39)$$

where $r(z)$ and $t(z)$ are entire functions and k is a natural number. From (37) and (39), we have

$$S[r(e^{z/k}) + t(e^{-z/k})] = e^c \cdot P(h(e^z))$$

so that

$$S[r(z) + t(1/z)] = e^c \cdot P(h(z^k)). \quad (40)$$

Now the right hand side of (40) is entire. Since $S(z)$ is a non-constant entire function, it follows from (40) that

$$t(z) \equiv \text{const.} = c' \text{ (say).} \quad (41)$$

Then the relation (40) can be written as

$$S[r(z) + c'] = e^c \cdot P(h(z^k)). \quad (42)$$

Also from (39) and (41), we have

$$g(z) = r(e^{z/k}) + c' = [r(z) + c'] \circ (e^{z/k}). \quad (43)$$

Further, from (38) and (43),

$$A[r(e^{z/k}) + c'] = Q(h(e^z)) - c$$

so that

$$A(r(z) + c') = Q(h(z^k)) - c. \quad (44)$$

Therefore, putting

$$U(z) = r(z) + c',$$

from (26), (42), (43) and (44), we've obtained

$$\begin{aligned} f(z) &= S(z) \cdot e^{Ac^z}, \\ g(z) &= [U(z)] \circ (e^{z/k}), \end{aligned}$$

such that

$$\begin{aligned} S(U(z)) &= e^c \cdot P(h(z^k)), \\ A(U(z)) &= Q(h(z^k)) - c \end{aligned}$$

where c is a constant, k is a natural number, $A(z)$, $S(z)$ and $U(z)$ are some entire functions. These correspond to (6), (7), (8) and (9) respectively.

Well, we wish to prove that the case where

$$B(z) \neq \text{const. (polynomial)} \quad (45)$$

leads us to a contradiction. Then our proof of Theorem 1 is completed.

Assume that (45) is valid. Then, applying Lemma 8 to (28) (under (30)), we have

$$A(z) = az + b \quad (a \neq 0); \text{ linear.} \quad (46)$$

Hence the identity (28) can be written as

$$g(z) = 1/a \cdot [Q(h(e^z)) - B(z) - b] \quad (47)$$

so that (29) becomes

$$S(1/a \cdot [Q(h(e^z)) - B(z) - b]) = P(h(e^z)) \cdot e^{B(z)}. \quad (48)$$

Let's compare the growth of the both sides of (48) along the imaginary axis. Note that $g(z)$ is continuously unbounded and of polynomial growth there. Further note that (31) holds, even if $S(z)$ is transcendental. Hence, by using the minimum modulus theorem for entire functions of order less than $1/2$ or Wiman's theorem (if necessary), we conclude that the left hand side of (48) is unbounded and at most of order-zero like growth along the imaginary axis. While, the right hand side is either bounded or at least of order-one like growth on the (upper or lower part of the) imaginary axis, according as $\exp[B(z)]$ is so. Hence the relation (48) cannot hold identically, a contradiction.

Thus we have the conclusion.

4. In this section, we'll note the following fact.

Theorem 2. *Let $h(z)$ be a non-constant entire function with $\rho(h)=0$ such that $h(0) \neq 0$. Then we have necessarily*

$$\rho^*(h(e^z)) = \rho(h(e^z)). \quad (49)$$

For the proof of this fact, we shall need the following result due to Goldstein, along with Lemmas 5, 6 and 7.

Lemma 9. (Goldstein [5]) *Let $F(z)$ be an entire function of finite order such that $\delta(a, F)=1$ for some $a \neq \infty$, where $\delta(\cdot, F)$ denotes the Nevanlinna deficiency. Then $F(z)$ is pseudo-prime.*

Proof of Theorem 2. If h is a polynomial, then the assertion (49) is clearly valid. Hence we may assume that h is transcendental.

Assume now that

$$\rho^*(h(e^z)) < \infty. \quad (50)$$

Then by Lemma 5, we have $h \in \underline{\mathbf{H}}$ so that by Lemma 6

$$\underline{\rho}(h(e^z)) < \infty. \quad (51)$$

Setting

$$h(e^z) = \Pi(z) \cdot e^{L(z)}, \quad (52)$$

where $\Pi(z)$ is the canonical product constructed from the zeros of $h(e^z)$ and $L(z)$ is an entire function, and noting

$$\rho(\Pi) = \rho^*(h(e^z)),$$

in view of (50), (51) and (52), we have

$$\underline{\rho}(e^{L(z)}) < \infty.$$

Hence by Lemma 7 we have that $L(z)$ is a polynomial so that by (52) we conclude

$$\rho(h(e^z)) < \infty. \quad (53)$$

Thus, in this case, (50) implies (53).

Now, under (53), assume further that

$$\rho^*[h(e^z)] < \rho[h(e^z)]. \quad (54)$$

Then for the representation such as (52), we see

$$\rho(0, \Pi e^L) = 1.$$

Note that $\rho(\Pi e^L) < \infty$. Hence, by Lemma 9, we know that the function $\Pi(z) \cdot e^{L(z)} = h(e^z)$ is pseudo-prime. However h (by the assumption) and e^z are both transcendental. Thus we have a contradiction. Therefore the assertion (49) must hold.

Note that if the left hand side of (49) is infinite, then also the right hand side is infinite.

From the proof of Theorem 2, we know that the following fact is valid.

Theorem 2'. *Let $f(z)$ be a transcendental entire function with $\rho(f)=0$ and $g(z)$ be an entire function with $\rho(g) < \infty$ and $\underline{\rho}(g) > 0$. Then we have*

$$\rho^*[f(g(z))] = \rho[f(g(z))].$$

Here, we'll consider the following **example** (cf. Titchmarsh, The theory of functions, 2nd ed. Oxford Univ. Press, 1939, p. 285, misc. ex. 10):

For $\alpha > 0$, let

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{e^z}{e^{n^\alpha}} \right) = h(e^z), \quad (55)$$

where

$$h(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{e^{n^\alpha}} \right). \quad (56)$$

Then we have

$$\rho^*(f) = \rho(f) = 1 + \frac{1}{\alpha}. \quad (57)$$

It is easily shown that h is of order zero. In fact: Since h has only negative zeros, for positive r , we obtain

$$\log M(r, h) = \log h(r) = r \cdot \int_0^\infty \frac{n(t)}{t(r+t)} dt$$

(cf. Titchmarsh, loc. cit., p. 271), where $n(t)$ denotes the number of the points z (including multiplicities) such that $h(z) = 0$ and $|z| \leq t$. In this case,

$$n(t) \sim [\log t]^{1/\alpha} \quad (t > t_0).$$

Hence

$$\begin{aligned} \log M(r, h) &\leq n(r) \cdot \int_1^r \frac{dt}{t} + r \cdot \int_r^\infty \frac{n(t)}{t^2} dt \\ &\leq K[(\log r)^{1+1/\alpha} + r \cdot \int_r^\infty \frac{(\log t)^{1/\alpha}}{t^2} dt] \\ &= K(1 + o(1)) (\log r)^{1+1/\alpha} \end{aligned}$$

for some constant K . The last assertion is shown by integrating partially the second term of the above [...].

Thus

$$\log M(r, h) \leq 2K (\log r)^{1+1/\alpha} \quad (r > r_0).$$

Further

$$\log M(r, f) = \log M(r, h(e^z)) \leq 2K \cdot r^{1+1/\alpha}$$

so that we have

$$(\rho^*(f) \leq) \rho(f) \leq 1 + 1/\alpha. \quad (58)$$

Next we shall prove that

$$\rho^*(f) = 1 + 1/\alpha. \quad (59)$$

At first, the set of the zeros of $f(z)$ is identical with the union of the set of the zeros of the function; $1 + e^z/e^{n^\alpha}$ for $n \geq 1$. Now the zeros of the above function are given by

$$z = n^\alpha + (2k+1)\pi i \quad (k; \text{integer}).$$

We shall compute the exponent of convergence of the serie:

$$T = \sum_{\substack{n \in \mathbb{N} \\ k \in \mathbb{Z}}} \frac{1}{[n^{2\alpha} + (2k+1)^2 \pi^2]^{s/2}}. \quad (60)$$

Setting

$$T' = \sum_{\substack{n \in N \\ k \in N}} \frac{1}{(n^{2\alpha} + k^2)^{s/2}}, \quad (61)$$

it follows

$$\frac{1}{(2\pi)^s} T' \leq T \leq 2T'. \quad (62)$$

Hence it suffices to consider the serie T' and so

$$T'' = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{(n^{2\alpha} + k^2)^{s/2}} \right), \quad (63)$$

since T' consists of only positive terms.

Here

$$\sum_{k=1}^{\infty} \int_1^{\infty} \frac{dx}{(x^{2\alpha} + k^2)^{s/2}} \leq T'' \leq \sum_{k=1}^{\infty} \int_0^{\infty} \frac{dx}{(x^{2\alpha} + k^2)^{s/2}}. \quad (64)$$

Well, as is easily shown, we have

$$\int_0^{\infty} \frac{dx}{(x^{2\alpha} + k^2)^{s/2}} = A(\alpha, s) \cdot \frac{1}{k^{s-1/\alpha}} \quad (65)$$

and

$$\int_1^{\infty} \frac{dx}{(x^{2\alpha} + k^2)^{s/2}} > B(\alpha, s) \cdot \frac{1}{k^{s-1/\alpha}}, \quad (66)$$

where $A(\alpha, s)$ and $B(\alpha, s)$ are some positive constants depending only on α and s such that they are finite if $s > 1/\alpha$ and infinite if $s \leq 1/\alpha$.

Indeed, they are taken as

$$A(\alpha, s) = \int_0^{\infty} \frac{dy}{(1 + y^{2\alpha})^{s/2}}$$

and

$$B(\alpha, s) = \int_1^{\infty} \frac{dy}{(1 + y^{2\alpha})^{s/2}}.$$

Hence we have

$$B(\alpha, s) \cdot \sum_{k=1}^{\infty} \frac{1}{k^{s-1/\alpha}} \leq T'' \leq A(\alpha, s) \cdot \sum_{k=1}^{\infty} \frac{1}{k^{s-1/\alpha}}.$$

While, the serie $\sum [1/k^{s-1/\alpha}]$ converges if and only if $s-1/\alpha > 1$, that is, $s > 1+1/\alpha$. Therefore the exponent of convergence for T'' (and so T) is equal to $1+1/\alpha$. Thus

$$\rho^*(f) = 1+1/\alpha,$$

which is to be shown.

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