# Analysis of Pulse Modulated Control Systems (I)

## Optimal Controls of Systems with Pulse Frequency Modulation and with Combined Pulse Modulation

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*Received May 1, 1978*

## 1. Introduction

The study of systems with pulse frequency modulation (PFM) has been the subject of many recent papers which were concerned either with general theory,[6~12] the engineering applications (attitude control of space vehicles,[3~5] etc.) or the applications in modeling of nervous system.[1,2]

Pulse frequency modulation may be divided into two broad classes in accordance with the mechanism of pulse emission. One class of PFM is so-called discrete PFM (or PFM of the first kind) in which the next pulse emission time is defined as the value of some nonlinear function of the input signal at the instant of present pulse emission. The other class of PFM is so-called dynamic PFM (or PFM of the second kind) where the pulse emission time is defined as the time when the value of some function of the input signal reaches a certain level after the present pulse emission. Among dynamic PFM's integral PFM (IPFM) and neural PFM (NPFM) are most extensively studied. The modulator of IPFM is defined as a device that emits a pulse whenever the absolute value of the integral of its input reaches a level and then it resets the integrator. In the NPFM modulator, the input signal is fed to a linear first order low pass filter and a pulse is emitted whenever the absolute value of the output of the filter reaches a level.

As for combined pulse modulation with pulse frequency-pulse width modulation, discrete PFM-PWM, IPFM-PWM and NPFM-PWM have also been proposed and discussed.[13~15]

In this paper, a new kind of pulse frequency modulation scheme referred to here as rate sensitive neural PFM (R. S. NPFM)[22,23] which is a generalization of NPFM and a combined pulse modulation scheme with R. S. NPFM law and pulse width modulation law referred to here as rate sensitive neural PFM-PWM (R. S. NPFM-PWM) are presented and discussed.

The major part of this paper is concerned with the study of optimum control problems for systems with R. S. NPFM and for systems with R. S. NPFM-PWM using nonlinear approximation approaches.[24,25]

Examples with experimental verification to justify the use of the method are also presented.

## 2. Definition and Description of R.S. NPFM Modulator

Fig. 1 shows schematic diagram of the modulator of R. S. NPFM with rectangular pulse output which is the actual form of the one of R. S. NPFM with impulse output[22] presented previously by the author.

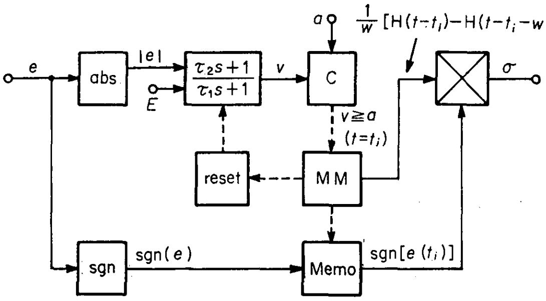


**Fig. 1** Rate sensitive neural PFM

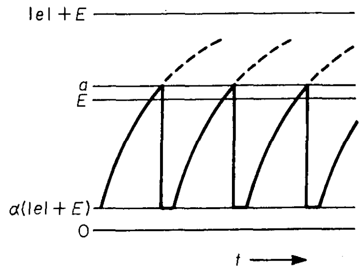


**Fig. 2** Waveform $v(t)$ in the R. S. NPFM to constant input

The modulator behaves as follows: Both the absolute value of input signal $e$ of the modulator and internal positive bias $E$ (which is smaller than threshold $a$) are applied to a linear element with transfer function

$$\left.\begin{aligned} G_p(s) &= \frac{\tau_2 s+1}{\tau_1 s+1} = \alpha + \frac{1-\alpha}{\tau_1 s+1} \\ \alpha &= \tau_2/\tau_1 \ (<1) \end{aligned}\right\} \tag{1}$$

where $\tau_1$ and $\tau_2$ are time constants.

At the instant when the output $v$ of the linear element reaches threshold $a$, comparator $C$ acts so that monostable-multibibrator MM generates a unit area rectangular pulse (of pulse width $w$ and pulse height $1/w$). The linear element is kept reset during the pulse emission. (Reset time may be defined to be infinitesimal with slight changes in Eqs. (3) and (5).). Thus, the $i$-th pulse emission time $t_i$ is defined as

$$t_i = \min\,[t : t \geq t_{i-1}+w,\ v(t) \geq a] \tag{2}$$

where

$$v(t) = \alpha(|e(t)|+E) + \frac{1-\alpha}{\tau_1}\int_{t_{i-1}+w}^{t} \exp\left(\frac{\tau - t}{\tau_1}\right)(|e(\tau)|+E)\,d\tau \tag{3}$$

The unit area rectangular pulse generated by the MM is so processed that the output pulse $\sigma$ of the modulator has the same sign as the input signal at each pulse emission time $t_i$ ($i=1, 2, \ldots$).

Thus,

$$\sigma(t)=\sum_i \operatorname{sgn}[e(t_i)]\frac{1}{w}[H(t-t_i)-H(t-t_i-w)] \tag{4}$$

where $H(t)$ is the Heaviside unit step function.

If we set $E=0$ and $\tau_2=0$ in the modulator, we have a modulator equivalent to that of conventional NPFM. The modulator, obtained from R. S. NPFM by setting $0<E<a$ and $\tau_2=0$, referred to here as NPFM with bias, has such an significant advantage over NPFM that, by virtue of bias $E$, the dead zone of the modulator can be made as small as desired without changing threshold value, or in other words, its threshold can be chosen large enough independently of the dead zone so that the modulator gets much better noise immunity. Adding a small differential term $\tau_2\, s$ to the linear element of the modulator of NPFM with bias, we have the one of R. S. NPFM which has quicker transient characteristic than the former.

If the input to the modulator is constant, i.e. $e=e_0$(=const.), it is easy to calculate the pulse interval $t_0$ from Eqs. (2) and (3), hence, the number of pulse emitted per unit time, i.e. pulse frequency, $f_0(e_0)$ is

$$f_0(e_0)=\frac{1}{t_0}=\begin{cases}\dfrac{1}{\tau_1}\cdot\dfrac{1}{\ln\dfrac{(1-a)(|e_0|+E)}{|e_0|+E-a}+\dfrac{w}{\tau_1}}, & (|e_0|>\Delta)\\ 0, & (|e_0|\leq\Delta)\end{cases} \tag{5}$$

where $\Delta$ is the dead zone of R. S. NPFM modulator,

$$\Delta=a-E \tag{6}$$

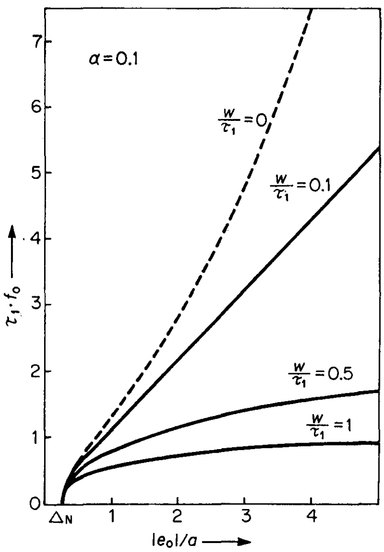


**Fig. 3** Static characteristics of number of emitted pulses of R. S. NPFM per unit time to constant input

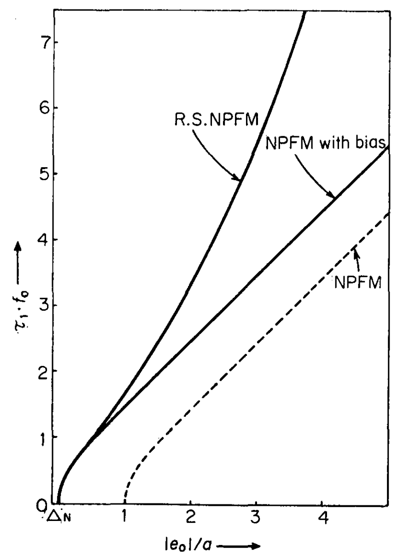


**Fig. 4** Comparison of static characteristics of various NPFM's

while the dead zone of NPFM modulator $\Delta = a$. The relation represented in Eq. (5) is plotted in Fig. 3 on normalized scales, where $\Delta_N = \Delta/a = (a-E)/a$ is the normalized dead zone. As $f_0$ is always smaller than $1/w$, each solid line with parameter $w/\tau_1$ approaches to the horizontal line $\tau_1/w$ as $|e_0|/a$ increases. The broken line with parameter $w/\tau_1 = 0$ indicates the characteristic of R. S. NPFM with impulse output. Static characteristics of NPFM, NPFM with bias and R. S. NPFM are shown in Fig. 4, to see the differences clearly, in the case where the output pulses of these NPFM's are all impulses. They are obtained from Eq. (5) setting $w=0$ for R. S. NPFM, $w=0$ and $a=0$ for NPFM with bias and $w=0$, $a=0$ and $E=0$ for NPFM. It will be noted in the figure that the characteristic of NPFM with bias is parallel to that of NPFM and the dead zone of NPFM with bias is reduced to $\Delta_N$ by the bias, which means that NPFM with bias can be made to respond to much smaller input signal than NPFM does. It will be also noted that R. S. NPFM has quicker response than NPFM with bias. Responses of these NPFM's and integrals of their outputs to rectangular and sinusoidal inputs are shown in Fig. 5.

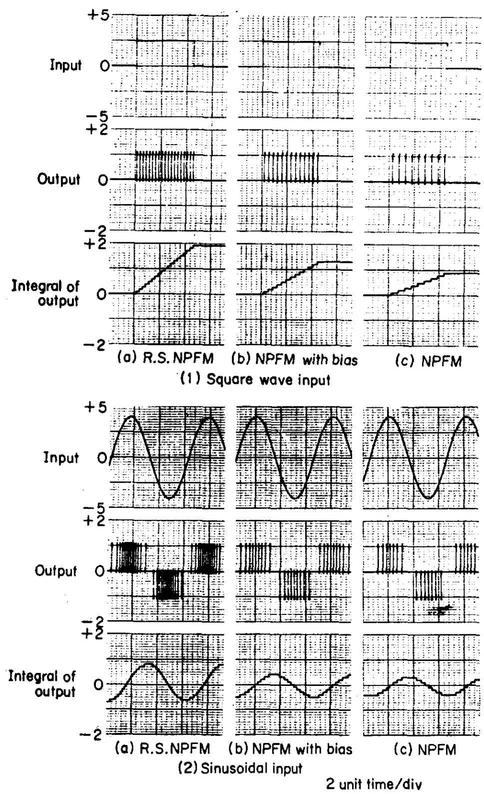


**Fig. 5** Responses of R. S. NPFM, NPFM with bias and NPFM to square wave and sinusoidal inputs

## 3. Definition and Description of R.S. NPFM-PWM Modulator

Fig. 6 shows the schematic diagram of the modulator of R. S. NPFM-PWM which is the combined modulation scheme with R.S. NPFM law and ordinary pulse width modulation law.

The $i$-th pulse emission time $t_i$ of the combined modulator is decided similarly as in R. S. NPFM, that is, $t_i$ is the first time when $v(t)$ is equal to or greater than threshold $a$ after

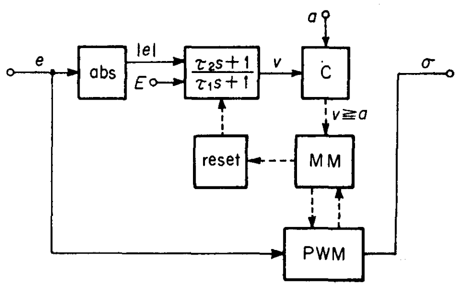


**Fig. 6** Rate sensitive neural PFM-PWM

the time $t_{i-1}+T_m$, where $T_m$ denotes minimum pulse interval. Thus, $t_i$ is defined as

$$t_i=\min\left[t:\ t\geq t_{i-1}+T_m,\ v(t)\geq a\right] \tag{7}$$

where

$$v(t)=\alpha(|e(t)|+E)+\frac{1-\alpha}{\tau_1}\int_{t_{i-1}}^{t}\exp\left(\frac{\tau-t}{\tau_1}\right)(|e(\tau)|+E)d\tau \tag{8}$$

where $\tau_1$, $\tau_2$: time constants ($\tau_2/\tau_1=\alpha<1$); $a$: threshold; $E$: positive bias ($<a$). Note that reset time is defined to be infinitesimal. The pulse width $w_i$ of the $i$-th pulse is defined to be proportional to the magnitude of the input signal at $t_i$, that is,

$$w_i=\min\left[k|e(t)|,\ w_M\right] \tag{9}$$

where $k$ is a positive constant and $w_M$ denotes maximum pulse width which must be not greater than $T_m$, i.e. $w_M\leq T_m$, to avoid overlapping of output pulses. Thus, the output of the modulator is

$$\sigma(t)=\sum_i \operatorname{sgn}\left[e(t_i)\right]h\left[H(t-t_i)-H(t-t_i-w_i)\right] \tag{10}$$

where $h$ denotes pulse height.

If $e=e_0$ (=const.), the pulse interval $t_0$ and the pulse width $w_0$ are given as

$$t_0=\max\left[\tau_1\cdot\ln\frac{(1-\alpha)(|e_0|+E)}{|e_0|+E-a},\ T_m\right] \tag{11}$$

$$w_0=\min\left[k|e_0|,\ w_M\right] \tag{12}$$

If we choose $T_m$ equal to the pulse interval of $|e_0|=e_M$ in the case that $e_M$ is given, and if we choose $w_M$ and $k$ to satisfy the following equation

$$\begin{aligned}T_m&=\tau_1\ln\frac{(1-\alpha)(e_M+E)}{e_M+E-a}\\&=w_M=ke_M\end{aligned} \tag{13}$$

then, the product $F_0(e_0)$ of the number of output pulses per unit time $P_0$ (=$1/t_0$) and the pulse area is given as

$$F_0(e_0)=P_0\cdot h\cdot w_0=\begin{cases}0, & (|e_0|\leq\Delta)\\ \dfrac{hk|e_0|}{\tau_1\ln\dfrac{(1-\alpha)(|e_0|+E)}{|e_0|+E-a}}, & (\Delta<|e_0|<e_M)\\ h, & (e_M\leq|e_0|)\end{cases} \tag{14}$$

where $\Delta=a-E$ is the dead zone of the combined modulator, and $F_0(e_0)$ is considered to indicate static characteristic of the combined modulator. Figs. 7 and 8 shows relations of Eqs. (11) to (14).

The combined pulse modulator has advantages of PFM and PWM which will be exhibited especially in the case when the modulator is incorporated in a feedback control system (PFWM feedback system). Comparison of responses of PFM feedback system with R. S. NPFM (shown in Fig. 9) and PFWM feedback system with R. S. NPFM-PWM (shown in Fig. 10) to step input and to sinusoidal input are shown in Figs. 11 and 12, respectively, where the transfer function of linear system is $G(s)=0.1/s\cdot 1/(0.5s+1)$ and modulator parameters are $\tau_1=1$, $a=1$, $E=0.9$, $\alpha=0.1$; $h=200$, $e_M=5$ and $k=0.01607$. It is observed in Figs. 11 and 12 that PFWM feedback system follows up more rapidly to the system input than PFM feedback system does.

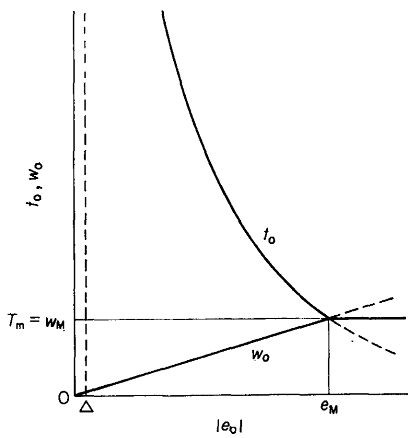


**Fig. 7** Relations of $t_0$ and $w_0$ to input $|e_0|$

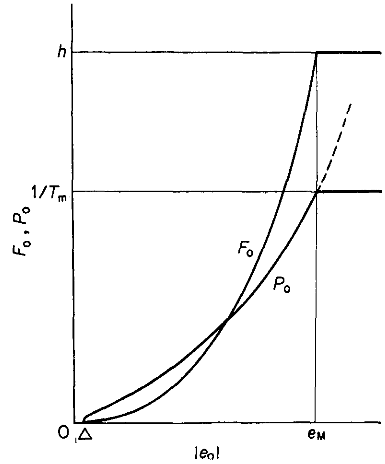


**Fig. 8** Relations of $F_0$ and $P_0$ to input $|e_0|$

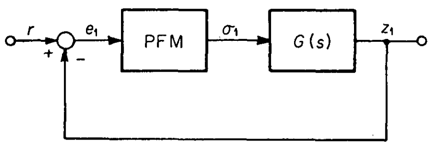


**Fig. 9** A PFM feedback control system

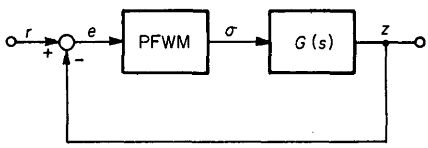


**Fig. 10** A PFWM feedback control system

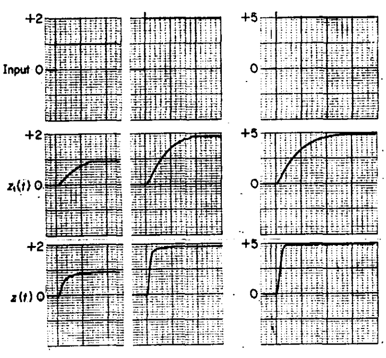


**Fig. 11** Step responses of PFM system and PFWM system

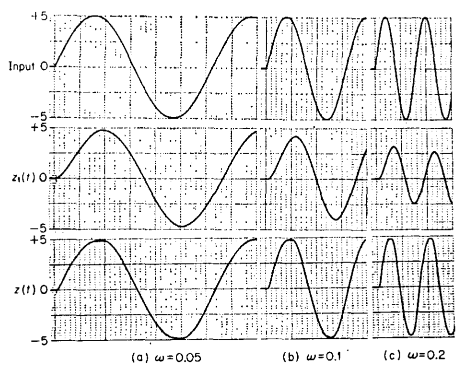


**Fig. 12** Responses of PFM system and PFWM system to sinusoidal input

## 4. Optimal Control of PFM systems

Optimum control problems of PFM system shown in Fig. 13 are considered here.

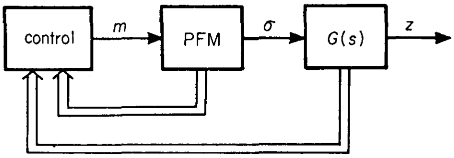


**Fig. 13** A PFM control system

The system is described as follows:

$$\dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+\boldsymbol{b}\sigma(t), \tag{15}$$

$$z(t)=\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}(t), \tag{16}$$

$$\sigma(t)=\sum_i \operatorname{sgn}\,[m(t_i)]\frac{1}{w}[H(t-t_i)-H(t-t_i-w)], \tag{17}$$

$$\left.\begin{aligned} &t_i=\min\,[t : t>t_{i-1}+w,\ v(t)\geq a], \\ &v(t)=\alpha(|m(t)|+E)+\frac{1-\alpha}{\tau_1}\int_{t_{i-1}+w}^{t}\exp\left(\frac{\tau-t}{\tau_1}\right)(|m(\tau)|+E)d\tau \end{aligned}\right\} \tag{18}$$

where $\boldsymbol{x}$ is the n-dimensional state vector of the linear system whose transfer function is $G(s)$, $A$ is $\mathrm{n}\times\mathrm{n}$ constant coefficient matrix, $\boldsymbol{b}$ and $\boldsymbol{d}$ are n-dimmensional constant vectors, $z$ is system output and $m$ is control variable.

Unfortunately, to the author's knowledge, there have been so far presented no mathematically exact method to solve optimum control problems for dynamic PFM system except in a few limited cases.[18] More precisely, although the pulse emission time $t_i$ is defined implicitly as a function of $m(t)$ by Eq. (18), the time $t_i$ cannot be expressed explicitly to arbitrary $m(t)$ except $m$ being constant, hence, the input-output relation of PFM modurator to arbitrary $m(t)$ cannot be determined analytically. Therefore, we are forced to use some approximation approach for the the problem.

First, let us assume that the transfer function $G(s)$ of the linear system be, as shown in Fig. 14,

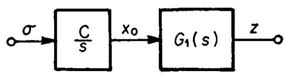


**Fig. 14** A controlled linear system $G(s)$

$$G(s)=\frac{\mathrm{C}}{s}\cdot G_1(s) \tag{19}$$

where C is the gain constant of the integrator which behaves as a demodulator of pulse frequency modulation and $G_1(s)$ is a linear subsystem. In this case, output $x_0(t)$ of the integrator is

$$x_0(t)=\mathrm{C}\int_0^t \sigma(t)dt+x_0(0) \tag{20}$$

Second, assume that $m(t)$ varies so slowly that it might be considered to be constant during each pulse interval, that is, $m(t)$ can be considered approximately to be constant in the time $t_{i-1}+w \leq t \leq t_i$ for each $i$. Then, the output of the integrator can be approximated by, with errors less than the gain constant C,

$$x_0(t) \simeq C\int_0^t \mathrm{sgn}(m)\cdot f_0(m)dt + x_0(0) \triangleq x_0'(t) \tag{21}$$

where $f_0(m)$ is the static characteristic of R. S. NPFM and is

$$f_0(m)=\begin{cases} \dfrac{1}{\tau_1}\cdot \dfrac{1}{\ln\dfrac{(1-a)(|m|+E)}{|m|+E-a}+\dfrac{w}{\tau_1}}, & (|m|>\Delta) \\ 0, & (m\leq\Delta) \end{cases} \tag{22}$$

Fig. 15 shows relation of Eq. (21) in this case.

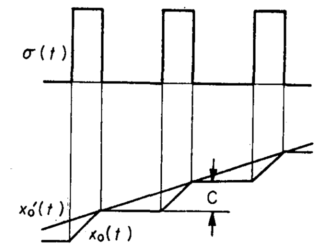


**Fig. 15** Input-output relation of the demodulating integrator of the PFM

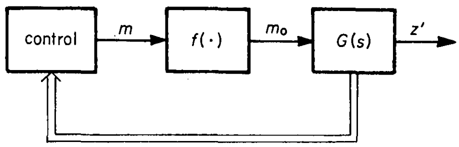


**Fig. 16** Approximated nonlinear system

Now, define a nonlinear function by

$$f(m)=\mathrm{sgn}(m)\cdot f_0(m) \tag{23}$$

which is considered to indicate an approximate input-output relation of the R. S. NPFM modulator. Then, define a nonlinear control system, shown in Fig. 16, whose nonlinear element $f(\cdot)$ has the characteristic of Eq. (23), as the approximated system to R. S. NPFM system shown in Fig. 13. Now, we can obtain approximate optimum control strategies for R. S. NPFM system, if we accept the optimum control strategies for the approximated system as the approximate ones for the PFM system.

Optimum control problems of the approximated system can be solved precisely and easily as follows. For $f(m)$ is a monotone single valued continuous function of $m$, $m_0(=f(m))$ is determined uniquely by $m$. Thus, to determine optimum control signal $m^*(t)$ of the approximated nonlinear system is equivalent to the determination of optimum control signal $m_0^*(t)$ of the linear system $G(s)$ subject to the constraint $|m_0|<1/w$, which will be done precisely and easily with well-known methods.[20,21] Here, define another nonlinear function, shown in Fig. 17, as

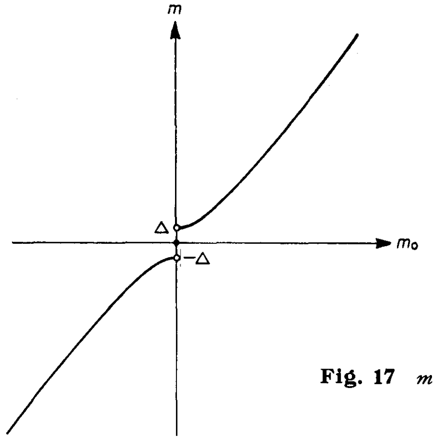


**Fig. 17** $m=l^{-1}(m_0)$

$$m=l^{-1}(m_0)=\begin{cases}0, & (m_0=0)\\ f^{-1}(m_0), & \left(m_0\neq 0,\ |m_0|<\dfrac{1}{w}\right)\end{cases} \tag{24}$$

then, $m_0^*$ corresponds to $m^*$ one-to-one. Thus, once $m_0^*$ is obtained, we have $m^*$ directly.

Fig. 18 shows an example of the transient responses of the approximated system and R. S. NPFM system to the optimum control signal $m^*(t)$ of the former calculated by analog computer. In this example, $G(s)=0.1/s\cdot 1/(2s+1)$; $a=1$, $E=0.9$, $\tau_l=1$, $\alpha=0.1$ and $w=0.1$; and $x_0'(t)$, $z'(t)$ denote the state variables of the approximated system. Three optimum control problems, (a) minimum time, (b) minimum fuel and (c) minimum energy problems, are examined to determine the control strategies such as to move the system from the initial state $x_0(0)=1$ and $z(0)=1$ to the final state $x_0(t_f)=0$ and $z(t_f)=0$ under the constraint $|m|\leq M=4$, minimizing the following criterion functions, respectively:

(a) $$J=\int_0^{t_f} dt$$

(b) $$J=\int_0^{5.4}|m|dt$$

(c) $$J=\int_0^{6}\frac{1}{2}(m^2+x_0^2+z^2)dt$$

It will be observed from the figure that responses of R. S. NPFM system to the optimal controls of the approximated system resemble closely to those of the approximated system. Examples of the same problems for R. S. NPFM system, NPFM with bias system and NPFM system are examined and shown in Figs. 19 to 21. The same problems as the above except the initial state being $x_0(0)=1$ and $z(0)=0$ are examined and shown in Figs. 22 to 24. These examples show the effectiveness of this approximation method.

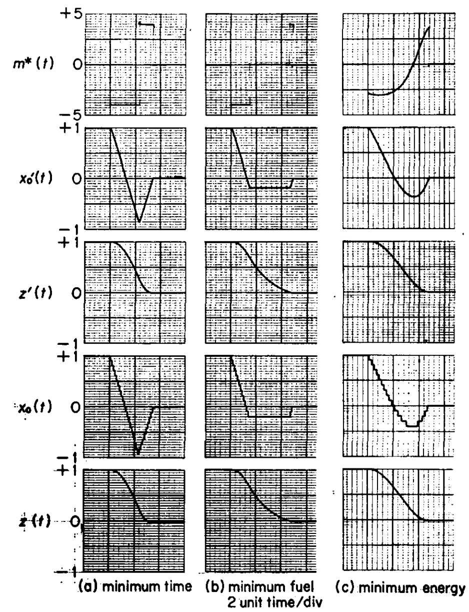


**Fig. 18** Transient responses of the approximated system and R. S. NPFM system to optimal control signal.

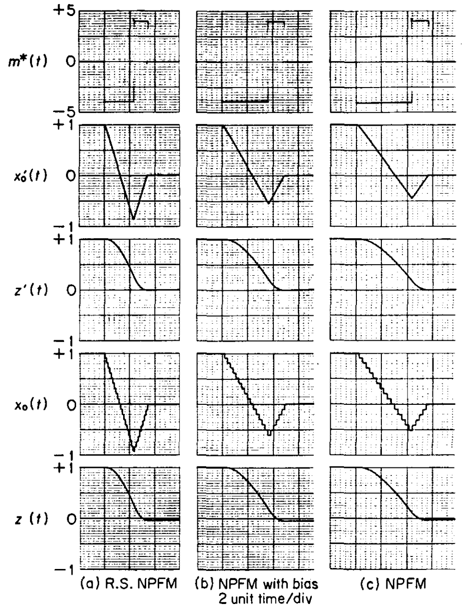


**Fig. 19** Transient responses of the approximated systems and PFM systems to 'minimum time' contro.l signals.

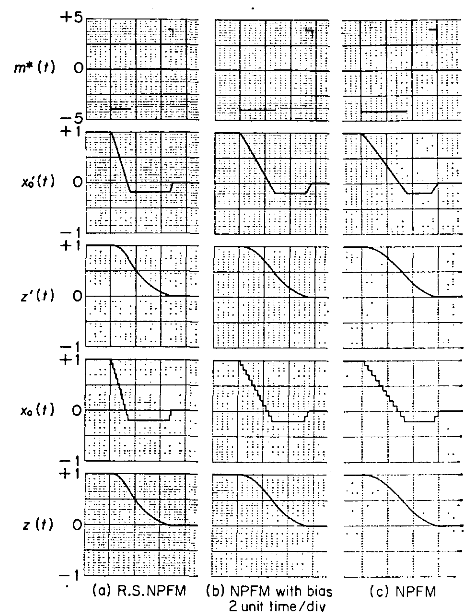


**Fig. 20** Transient responses of the approximated systems and PFM systems to 'minimum fuel' control signals.

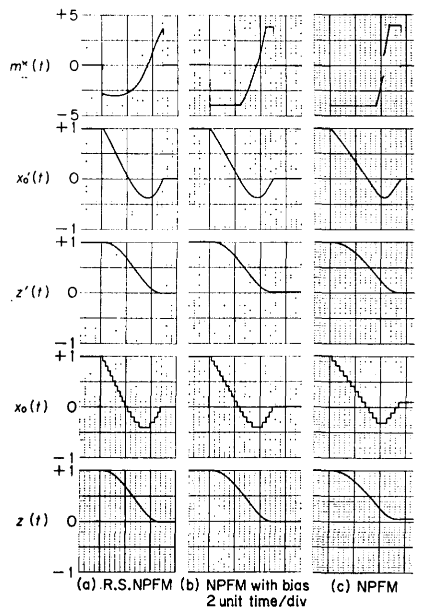


**Fig. 21** Transient responses of the approximated systems and PFM systems to 'minimum energy' control signals.

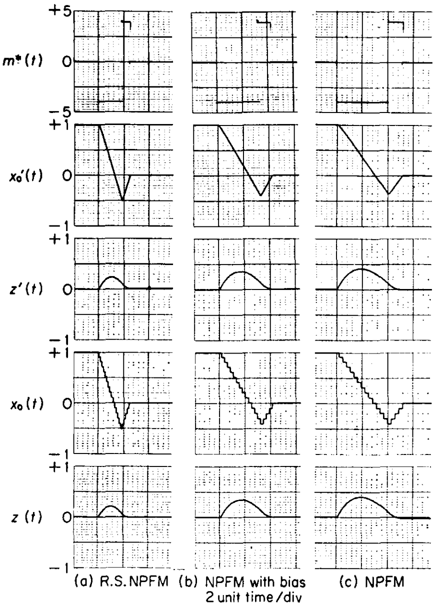


**Fig. 22** Transient responses of the approximated systems and PFM systems to 'minimum time' control signals.

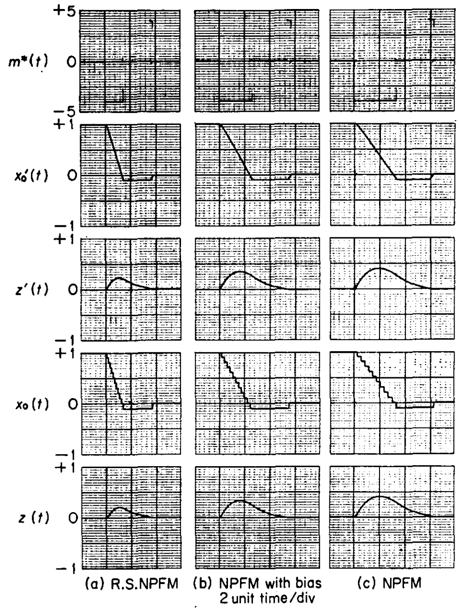


**Fig. 23** Transient responses of the approximated systems and PFM systems to 'minimum fuel' control signals.

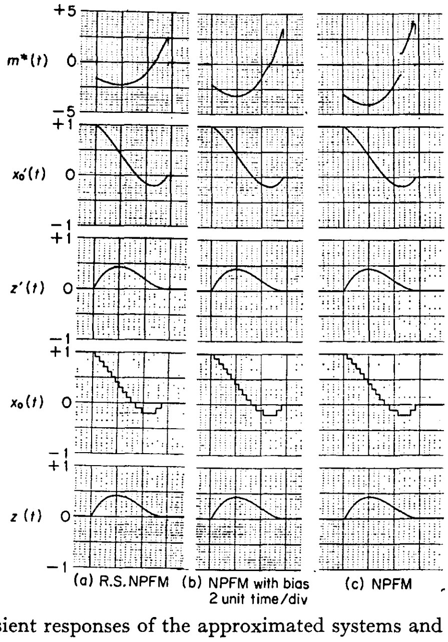


**Fig. 24** Transient responses of the approximated systems and PFM systems to 'minimum energy' control signals.

## 5. Optimal Control of R.S. NPFM-PWM system

The approximation method developed in the previous section is also applicable to optimum control problems of R. S. NPFM-PWM control system shown in Fig. 25.

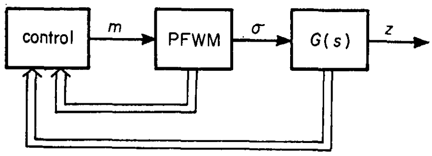


**Fig. 25** A PFWM control system

$$\dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+\boldsymbol{b}\sigma(t) \tag{24}$$

$$z(t)=\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}(t) \tag{25}$$

$$\sigma(t)=\sum_i \mathrm{sgn}\,[m(t_i)]h[\mathrm{H}(t-t_i)-\mathrm{H}(t-t_i-w_i)] \tag{26}$$

$$\left.\begin{aligned} &t_i=\min\,[t : t\geq t_{i-1}+T_m,\ v(t)\geq a] \\ &v(t)=a(|m(t)|+E)+\frac{1-a}{\tau_1}\int_{t_{i-1}}^{t}\exp\left(\frac{\tau-t}{\tau_1}\right)(|m(\tau)|+E)d\tau \end{aligned}\right\} \tag{27}$$

$$w_i=\min\,[k|m(t_i)|,\ w_M] \tag{28}$$

where the notations are same as before.

First, assume that $G(s)$ be

$$G(s)=\frac{\mathrm{C}}{s}\cdot G_1(s) \tag{30}$$

then, output $x_0(t)$ of the integrator is

$$x_0(t)=\mathrm{C}\int_0^t \sigma(t)dt+x_0(0) \tag{31}$$

Second, assume that $m(t)$ can be considered approximately to be constant during each pulse interval. Then, the output of the integrator can be approximated by, with errors less than the product of the gain constant of the integrator and one pulse area $C\cdot hk|m|$,

$$x_0(t)\simeq\mathrm{C}\int_0^t \mathrm{sgn}\,(m)\cdot F_0(m)dt+x_0(0)\triangleq x_0'(t) \tag{31}$$

where $F_0(m)$ is

$$F_0(m)=\begin{cases} 0, & (|m|\leq\Delta) \\ \dfrac{h\cdot k|m|}{\tau_1\ln\dfrac{(1-a)(|m|+E)}{|m|+E-a}}, & (\Delta<|m|<e_M) \\ h, & (e_M\leq|m|) \end{cases} \tag{32}$$

Now, define a nonlinear function by

$$F(m) = \mathrm{sgn}\,(m)\cdot F_0(m) \tag{33}$$

and define a nonlinear system, shown in Fig. 26, as the approximated system of R. S. NPFM-PWM system shown in Fig. 25. Next, define another nonlinear function as

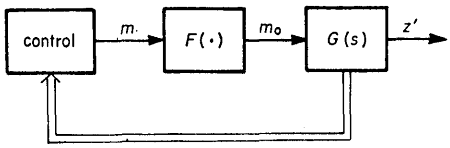


**Fig. 26** Approximated nonlinear system

$$m = L^{-1}(m_0) = \begin{cases} 0, & (m_0 = 0) \\ F^{-1}(m_0), & (m_0 \neq 0,\ |m_0| \leq h) \end{cases} \tag{34}$$

Once the optimal control $m_0{}^*$ of the linear system $G(s)$ under the constraint $|m_0| \leq h$, we have the optimal control $m^*$ of the approximated system directly which is the approximate one of R. S. NPFM-PWM system.

Fig. 27 shows an example of the transient responses of the approximated system and R. S. NPFM-PWM system to the minimum time controls of the former such that move the system from the three different initial states to the final state $x_0(t_f) = 0$ and $z(t_f) = 0$ in minimum

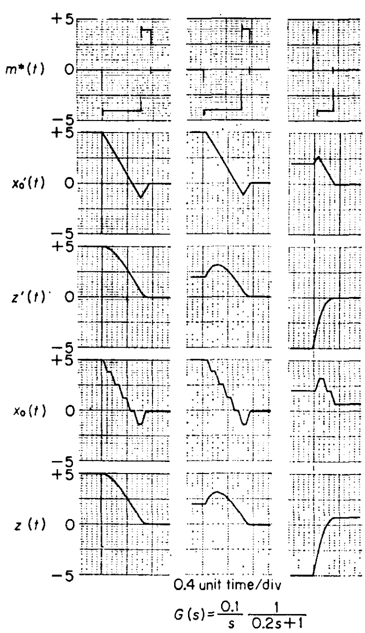


**Fig. 27** Transient responses of the approximated system and PFWM system to 'minimum time' control signals

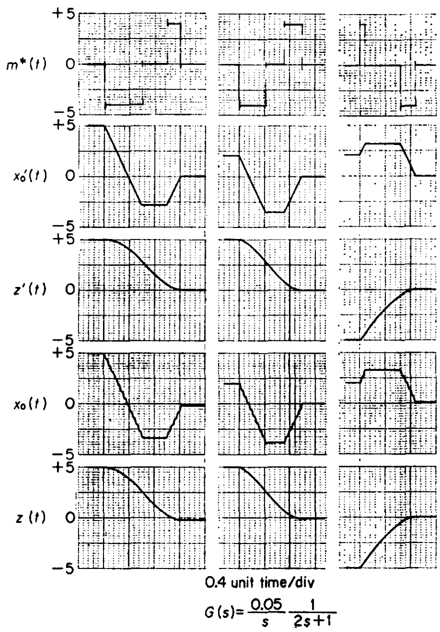


**Fig. 28** Transient responses of the approximated system and PFWM system to 'minimum fuel' control signals

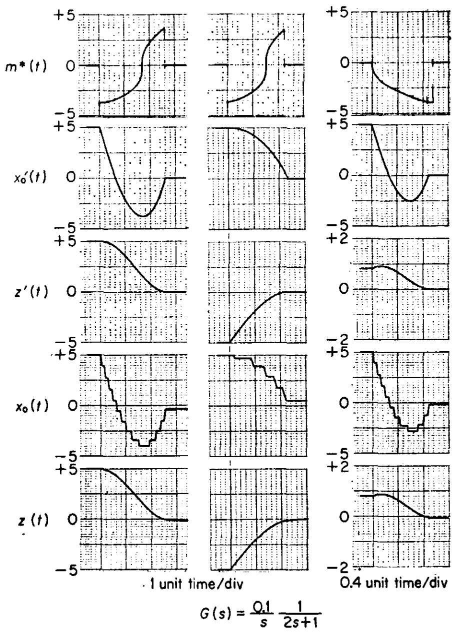


**Fig. 29** Transient responses of the approximated system and PFWM system to 'minimum energy' control signals

time under the constraint $|m|\leq M=4$, respectively. In this example, $a=1$, $E=0.9$, $\tau_1=1$, $\alpha=0.1$; $h=200$, $e_M=5$, $k=0.0167$. Figs. 28 and 29 show examples of minimum fuel and minimum energy control problems, respectively, where parameters are same as above. It will be observed that, in each case, response of PFWM system is similar to that of the approximated system with error less than $Chk|m|$.

## 6. Conclusion

A pulse frequency modulation scheme R. S. NPFM which is a generalization of conventional NPFM and a combined pulse frequency and width modulation scheme R. S. NPFM-PWM are presented and discussed in detail.

Optimum control problems for the systems incorporating R. S. NPFM, NPFM with bias and NPFM, respectively, are studied using the nonlinear functions which are the static characteristics of their modulators. Examples of optimum control problems of practical importance, (a) minimum time, (b) minimum fuel and (c) minimum energy problems, are examined with the results that the approximate optimal control moves PFM system with the errors less than C (i.e. C$\times$(one pulse area)) in each case. It is needless to say that the method is applicable to other optimal control problems not shown here. The approximation method is shown also to be applicable to optimum control problems for R. S. NPFM-PWM system and similar results are obtained.

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