Analysis of Pulse Modulated Control Systems (II)

Periodic Oscillations in Feedback Systems with Pulse Frequency Modulation and with Combined Pulse Modulation

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1. Introduction

Although systems with pulse frequency modulation (PFM) have such an wide range of applications from attitude control of spacecraft to modeling of nervous system,[1,2] many papers were concerned with the area of stability problem and comparatively few with the area of self-excited periodic oscillations and very few with the area of optimum control problems.

In the part (I) of this series of papers, a new kind of PFM referred to as R. S. NPFM which is a generalization of conventional neural PFM and a combined pulse modulation referred to as R. S. NPFM-PWM have been presented. Optimum control problems of practical importance for R. S. NPFM system and for R. S. NPFM-PWM system have been studied.

In this paper, self-excited periodic oscillations in the feedback system with R. S. NPFM and in the feedback system with R. S. NPFM-PWM are studied using specially developed quasi-describing functions,[10,11] respectively, since the conventional methods of analysis[3~8] are not applicable to these systems. Examples with experimental verification are also presented.

2. Periodic Oscillations in a Feedback System with Pulse Frequency Modulation

Let us consider the self-excited oscillations in the PFM feedback system shown in Fig. 1 incorporating R. S. NPFM.

2.1 Description of R.S. NPFM Modulator

The modulator of R. S. NPFM with rectangular pulse output is shown in Fig. 2 and is defined as follows:

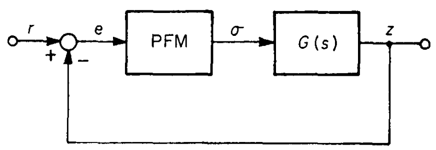

Fig. 1 A feedback system with a pulse frequency modulator.

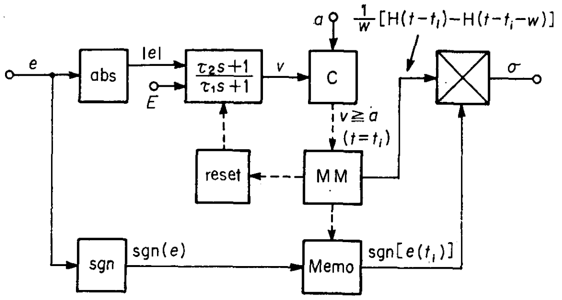

Fig. 2 Rate sensitive neural PFM

$$\sigma(t)=\sum_i \operatorname{sgn}[e(t_i)]\frac{1}{w}[\mathrm{H}(t-t_i)-\mathrm{H}(t-t_i-w)] \tag{1}$$

$$\left.\begin{aligned} &t_i=\min[t:\ t\geq t_{i-1}+w,\ v(t)\geq a] \\ &v(t)=\alpha(|e(t)|+E)+\frac{1-\alpha}{\tau_1}\int_{t_{i-1}+w}^{t}\exp\left(\frac{\tau-t}{\tau_1}\right)(|e(\tau)|+E)d\tau \end{aligned}\right\} \tag{2}$$

where e: modulator input; σ: modulator output; τ_1, τ_2: time constants ($\tau_2/\tau_1=\alpha<1$); a: threshold; E: positive bias ($<a$); w: pulse width. (For detailed description, see part (I) of this series.)

2.2 Analysis of Oscillations

Let us consider the oscillations in the feedback system shown in Fig. 1, for simplicity, in the case that the linear system has the transfer function

$$G(s)=\frac{\mathrm{C}}{s}\cdot\frac{\mathrm{K}}{s^2+\mathrm{F}s+\mathrm{K}} \tag{3}$$

where C is the gain constant of the integrator that behaves as a demodulator of pulse frequency modulation and F, K are system constants. The transfer function of second order system in $G(s)$ could be replaced by others as the occasion demands. The feedback system may fall into a symmetric or asymmetric self-excited oscillation depending on the system parameters and on the initial state.

2.2.1. Symmetric Oscillation (1)

When the system under consideration is in the state of periodic oscillation, the number N of positive pulses emitted is equal to that of negative ones in a period T. For the waveform of oscillation to be symmetric with respect to time axis, N must be even. Here, we assume the oscillation to be sinusoidal as

$$e(t)=A\sin\omega t \tag{4}$$

Since pulse emission time to the sinusoidal input cannot be expressed explicitly from Eq.

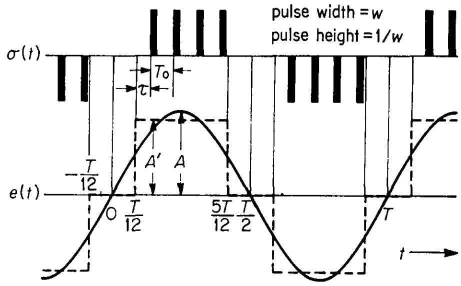

Fig. 3 Response of the PFM to an approximated sinusoidal wave (1)

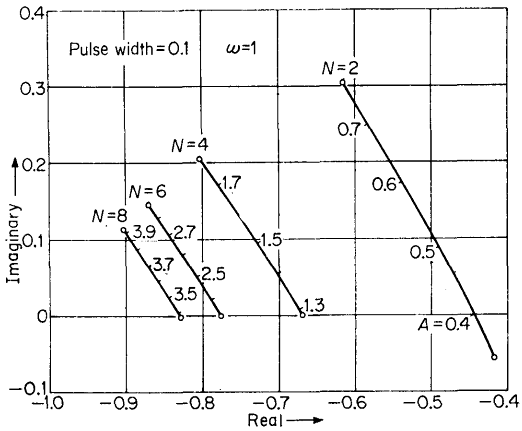

Fig. 4 Amplitude locus of $-1/G_{QD_1}$

(2), we approximate input $e(t)$ by a square wave $e'(t)$ shown by broken line in Fig. 3. The square wave is of height A' and of width $2\pi/3$. Setting $A'=(\pi/2\sqrt{3})A$, $e'(t)$ is expressed as

$$e'(t)=A\sin\omega t-\frac{1}{5}A\sin 5\omega t+\dots$$

For the low pass characteristic of the linear element in the modulator, higher harmonics of $e'(t)$ will be considered to affect little to the behavior of the modulator. As stated above, $e'(t)$ is defined in a period as

$$\left.\begin{array}{ll} e'(t)=0 & : 0\leq t\leq T/12 \\ e'(t)=A' & : T/12<t<5T/12 \\ e'(t)=0 & : 5T/12\leq t\leq 7T/12 \\ e'(t)=-A' & : 7T/12<t<11T/12 \\ e'(t)=0 & : 11T/12\leq t\leq T \end{array}\right\} \tag{5}$$

In this case, the output σ is, in one period,

$$\begin{aligned}\sigma(t)=&\sum_{i=1}^{N}\frac{1}{w}\left[\mathrm{H}\left[t-\left\{\frac{T}{12}+\tau+(i-1)T_0\right\}\right]-\mathrm{H}\left[t-\left\{\frac{T}{12}+\tau+(i-1)T_0+w\right\}\right]\right]\\ &-\sum_{i=1}^{N}\frac{1}{w}\left[\mathrm{H}\left[t-\left\{\frac{7T}{12}+\tau+(i-1)T_0\right\}\right]-\mathrm{H}\left[t-\left\{\frac{7T}{12}+\tau+(i-1)T_0+w\right\}\right]\right]\end{aligned} \tag{6}$$

where τ denotes the time lag of the first pulse and is obtained from (2), assuming $v(T/12)=v(7T/12)=E$, as

$$\tau=\tau_1\ln\frac{A'(1-a)}{A'+E-a} \tag{7}$$

and T_0 denotes the interval of successive output pulses and is obtained from (2), setting $|e'|=A'$, as

$$T_0=\tau_1 \ln \frac{(A'+E)(1-a)}{A'+E-a} \tag{8}$$

While the fundamental component $\sigma_1(t)$ of the Fourier series expansion of $\sigma(t)$ is

$$\sigma_1(t)=\frac{2\omega}{\pi}\cdot\frac{\sin\frac{\omega w}{2}}{\frac{\omega w}{2}}\cdot\frac{\sin\frac{N\omega T_0}{2}}{\sin\frac{\omega T_0}{2}}\sin\left\{\omega t+\frac{\pi}{3}-\omega\left(\tau+\frac{N-1}{2}T_0+\frac{w}{2}\right)\right\} \tag{9}$$

Thus, we have the quasi-describing function G_{QD_1} as

$$G_{QD_1}=\frac{2\omega}{\pi A}\cdot\frac{\sin\frac{\omega w}{2}}{\frac{w\omega}{2}}\cdot\frac{\sin\frac{N\omega T_0}{2}}{\sin\frac{\omega T_0}{2}}\exp\left[j\left\{\frac{\pi}{3}-\omega\left(\tau+\frac{N-1}{2}T_0+\frac{w}{2}\right)\right\}\right] \tag{10}$$

where $A=(2\sqrt{3}/\pi)A'$. The amplitude locus of $-1/G_{QD_1}$ is illustrated in Fig. 4, with parameters $a=1$, $E=0.9$, $\tau_1=1$, $\alpha=0.1$ and $w=0.1$. If the locus of $-1/G_{QD_1}$ intersects with that of $G(j\omega)$ for the same ω, and if the locus of $-1/G_{QD_1}$ crosses the point of intersection from the side of origin with respect to $G(j\omega)$ towards the other side as amplitude A increases, then the self-excited oscillation exists and is stable. Amplitude A and angular frequency ω of the oscillation and the number N of pulses per half period are determined from the point.

2.2.2. Symmetric Oscillation (2)

We assume again that self-exicted oscillation is of sinusoidal waveform and approximate this wave to a trapezoidal wave e'' (see the broken line in Fig. 5) with height A'. We also assume that N pulses are generated between τ and $T/2-\tau$; τ is the time lag of the first pulse and is determined by the following equation:

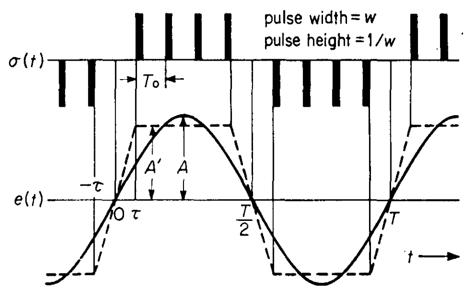

Fig. 5 Response of the PFM to an approximated sinusoidal wave (2)

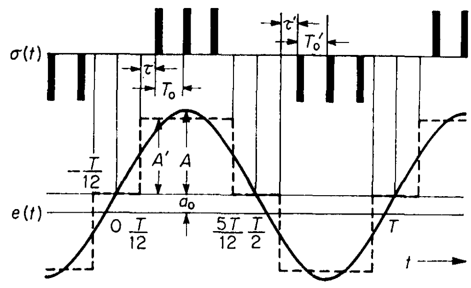

Fig. 6 Response of the PFM to an approximated sinusoidal wave (3)

$$\left.\begin{aligned} &2\tau+(N-1)T_0=T/2 \\ &T_0=\tau_1 \ln \frac{(A'+E)(1-a)}{A'+E-a}+w \end{aligned}\right\} \tag{11}$$

In this case, output $\sigma(t)$ is given by

$$\sigma(t)=\sum_{i=1}^{N}\frac{1}{w}[\mathrm{H}[t-\{\tau+(i-1)T_0\}]-\mathrm{H}[t-\{\tau+(i-1)T_0+w\}]]$$

$$-\sum_{i=1}^{N}\frac{1}{w}\left[\mathrm{H}\left[t-\left\{\frac{T}{2}+\tau+(i-1)T_0\right\}\right]-\mathrm{H}\left[t-\left\{\frac{T}{2}+\tau+(i-1)T_0+w\right\}\right]\right] \quad (12)$$

The fundamental component of $\sigma_1(t)$ is expressed as

$$\sigma_1(t)=\frac{2\omega}{\pi}\cdot\frac{\sin\frac{\omega w}{2}}{\frac{\omega w}{2}}\cdot\frac{\sin\frac{N\omega T_0}{2}}{\sin\frac{\omega T_0}{2}}\sin\omega t \quad (13)$$

Comparing $\sigma_1(t)$ with input $e(t)$, we have the quasi-describing function

$$G_{QD_2}=\frac{2\omega}{\pi A}\cdot\frac{\sin\frac{\omega w}{2}}{\frac{\omega w}{2}}\cdot\frac{\sin\frac{N\omega T_0}{2}}{\sin\frac{\omega T_0}{2}} \quad (14)$$

where the amplitude A of input is equal to the amplitude of the fundamental component of trapezoidal wave $e''(t)$, hence,

$$A=\frac{8A'}{\pi}\cdot\frac{\cos[\omega\{(N-1)T_0+w\}/2]}{\pi-\omega\{(N-1)T_0+w\}}$$

From Eq. (14), amplitude locus of $-1/G_{QD_2}$ always lies on the real axis. As stated previously, the point of intersection of loci of $-1/G_{QD_2}$ and $G(j\omega)$ gives the amplitude A and angular frequency ω of the oscillation and the number N of pulses per half period. In the present case, it is very easy to check whether or not the sustained oscillations exist, for the locus of $-1/G_{QD_2}$ lies only on the real axis.

2.2.3. Asymmetric Oscillation

An asymmetric oscillation of the following form is studied here:

$$e(t)=a_0+A\sin\omega t \quad (15)$$

In this case, the number N of pulses per half period may be even or odd.

We approximate input $e(t)$ to the square wave $e'(t)$ shown by the broken line in Fig. 6. In this case, $e'(t)$ is expressed in a period as

$$\left.\begin{aligned}
e'(t)&=a_0 &&: 0\leq t\leq T/12\\
e'(t)&=a_0+A' &&: T/12<t<5T/12\\
e'(t)&=a_0 &&: 5T/12\leq t\leq 7T/12\\
e'(t)&=a_0-A' &&: 7T/12<t<11T/12\\
e'(t)&=a_0 &&: 11T/12\leq t\leq T
\end{aligned}\right\} \quad (16)$$

and output $\sigma(t)$ is

$$\sigma(t)=\sum_{i=1}^{N}\frac{1}{w}\left[\mathrm{H}\left[t-\left\{\frac{T}{12}+\tau+(i-1)T_0\right\}\right]-\mathrm{H}\left[t-\left\{\frac{T}{12}+\tau+(i-1)T_0+w\right\}\right]\right]$$

$$-\sum_{i=1}^{N}\frac{1}{w}\Big[\mathrm{H}\Big[t-\Big\{\frac{7T}{12}+\tau'+(i-1)T_0'\Big\}\Big]-\mathrm{H}\Big[t-\Big\{\frac{7T}{12}+\tau'+(i-1)T_0'+w\Big\}\Big]\Big] \tag{17}$$

where, τ is the time lag of the first positive pulse after e' has become equal to a_0+A' and τ' is the time lag of the first negative pulse after e' has become equal to a_0-A'. τ and τ' are obtained from Eq. (2) as

$$\left.\begin{aligned}\tau&=\tau_1\ln\frac{A'(1-a)}{A'+a_0+E-a}\\ \tau'&=\tau_1\ln\frac{A'(1-a)}{A'-a_0+E-a}\end{aligned}\right\} \tag{18}$$

and pulse intervals of positive and negative pulses are given, respectively, by

$$\left.\begin{aligned}T_0&=\tau_1\ln\frac{(A'+a_0+E)(1-a)}{A'+a_0+E-a}\\ T_0'&=\tau_1\ln\frac{(A'-a_0+E)(1-a)}{A'-a_0+E-a}\end{aligned}\right\} \tag{19}$$

The fundamental component of $\sigma_1(t)$ is

$$\sigma_1(t)=\frac{\omega}{\pi}\cdot\frac{\sin\frac{\omega w}{2}}{\frac{\omega w}{2}}\sqrt{(A_1+A_2)^2+(B_1+B_2)^2}\,\sin\Big(\omega t-\frac{\pi}{6}+\tan^{-1}\frac{B_1+B_2}{A_1+A_2}\Big) \tag{20}$$

where

$$\left.\begin{aligned}A_1&=\frac{\sin\frac{N\omega T_0}{2}}{\sin\frac{\omega T_0}{2}}\sin\omega\Big(\tau+\frac{N-1}{2}T_0+\frac{w}{2}\Big)\\ A_2&=\frac{\sin\frac{N\omega T_0'}{2}}{\sin\frac{\omega T_0'}{2}}\sin\omega\Big(\tau'+\frac{N-1}{2}T_0'+\frac{w}{2}\Big)\\ B_1&=A_1\cot\omega\Big(\tau+\frac{N-1}{2}T_0+\frac{w}{2}\Big)\\ B_2&=A_2\cot\omega\Big(\tau'+\frac{N-1}{2}T_0'+\frac{w}{2}\Big)\end{aligned}\right\} \tag{21}$$

Thus, we have the quasi-describing function for this case,

$$G_{QD_3}=\frac{\omega}{\pi A}\,\frac{\sin\frac{\omega w}{2}}{\frac{\omega w}{2}}\sqrt{(A_1+A_2)^2+(B_1+B_2)^2}\,\exp\Big[j\Big(-\frac{\pi}{6}+\tan^{-1}\frac{B_1+B_2}{A_1+A_2}\Big)\Big] \tag{22}$$

where $A=(2\sqrt{3}/\pi)A'$. As before, the asymmetric oscillation can be analyzed to inspect the point of intersection of loci of $-1/G_{QD_3}$ and $G(j\omega)$. On the other hand, it is necessary to know the value of a_0 *apriori* for the plot of $-1/G_{QD_3}$. Considering that the value of a_0 is equal to the average of the output $x_0(t)$ of the integrator C/s in the linear system $G(s)$, and that $x_0(t)$ varies stepwise and takes its value equal to the multiple of C, if we assume $\tau \simeq \tau'$ and $T_0 \simeq T_0'$, we have

$$a_0 \simeq C/2 \cdot n \quad (n=1, 2, \ldots) \tag{23}$$

Further, it is necessary for pulse number N to be even if n is even and odd if n is odd.

Trapezoidal wave approximation is not applicable for the analysis of asymmetric ocillation.

2.2.3. Examples

Examples of self-excited oscillations in the feedback system calculated by analog computer are shown in Figs. 7 to 9. In these examples, system parameters are same as $a=1$, $E=0.9$, $\alpha=0.1$, $\tau_L=2$, $w=0.1$; $C=0.1$, $F=0.085$ and $K=0.465$; while the initial states of the system are different. Fig. 10 shows the regions of initial states leading to these self-excited oscillations

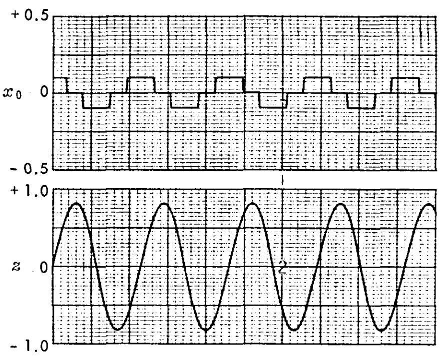

Fig. 7 4-pulse type self-excited symmetric oscillation.

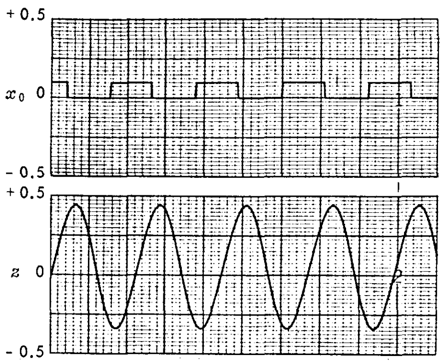

Fig. 8 2-pulse type self-excited asymmetric oscillation.

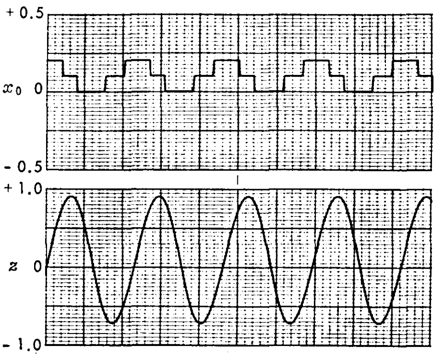

Fig. 9 4-pulse type self-excited asymmetric oscillation.

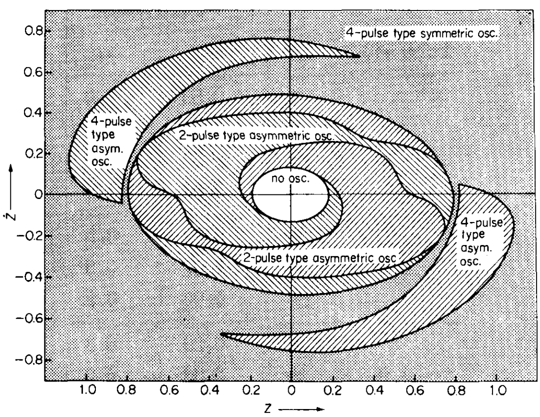

Fig. 10 Regions of initial states leading to various types of self-exchited periodic oscillation.

which is determined experimentally by analog computer. Analyses of these oscillations by the G_{QD} methods (a_0 is estimated from Eq. (23)) and the solutions obtained from Eqs. (1) to (3) by digital computer are shown in Tables 1 and 2. Tables 3 and 4 show the results of oscillations in the feedback system with different values of parameters other than a, E, α and C. It will be observed from these tables that the describing function method based on G_{DQ_1} gives good results when the amplitudes of oscillations are small. While the G_{QD_2} method gives a convenient tool for the analysis of symmetric oscillations. The G_{QD_3} method gives rather good results, while the value of a_0 is estimated from Eq. (23).

Table 1 Symmetric Oscillation in Fig. 7.

		G_{QD_1} method		G_{QD_2} method		digital computer	
Pulse width	$2N$	A	ω	A	ω	A	ω
0.1	4	0.765	0.695	0.791	0.682	0.836	0.689

Table 2 Asymmetric Oscillations in Figs. 8 and 9.

		G_{QD_3} method			digital computer		
Pulse width	$2N$	A	ω	a_0	A	ω	a_0
0.1	2	0.327	0.729	0.05	0.393	0.714	0.0491
	4	0.726	0.697	0.1	0.830	0.688	0.0983

Table 3 Symmetric Oscillations.

		G_{QD_1} method		G_{QD_2} method		digital computer	
Pulse width	$2N$	A	ω	A	ω	A	ω
0.1	4	0.55	1.0	0.61	0.98	0.579	0.994
	8	1.51	1.0	1.48	0.99	1.485	1.039
	12	2.53	1.0	2.34	1.0	2.434	1.001
	16	3.64	1.0	3.36	1.0	3.437	1.004
0.5	4	0.32	0.5	0.35	0.48	0.317	0.503
	8	1.07	0.5	1.05	0.5	1.013	0.506
	12	0.90	0.3	0.71	0.3	0.837	0.306
1.0	4	0.26	0.3	0.28	0.28	0.256	0.302
	8	0.38	0.2	0.35	0.19	0.358	0.205
	12	0.80	0.2	0.65	0.2	0.728	0.206

Table 4 Asymmetric Oscillations.

		G_{QD_3} method			digital computer		
Pulse width	$2N$	A	ω	a_0	A	ω	a_0
0.1	2	0.4	1.0	0.05	0.554	0.971	0.0495
	4	1.2	1.0	0.1	1.332	0.994	0.0998
	6	2.0	1.0	0.05	2.080	0.994	0.0492
	8	1.35	0.5	0.1	1.339	0.498	0.0978
	10	1.8	0.5	0.05	1.750	0.500	0.0494
0.5	2	0.25	0.5	0.05	0.282	0.491	0.0476
	4	0.7	0.5	0.1	0.719	0.499	0.0998
	6	1.5	0.5	0.05	1.382	0.504	0.0511
	8	1.45	0.4	0.1	1.421	0.400	0.0971
1.0	2	0.32	0.5	0.05	0.347	0.495	0.0487
	4	1.0	0.5	0.1	1.008	0.501	0.0995
	6	1.6	0.4	0.05	1.472	0.404	0.0510

3. Periodic Oscillations in a Feedback System with Combined Pulse Modulation

Let us consider the self-excited oscillations in the feedback system shown in Fig. 11 incorporating R. S. NPFM-PWM.

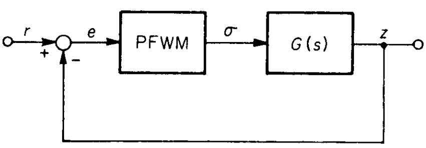

Fig. 11 A feedback system with a pulse frequency-pulse width modulator.

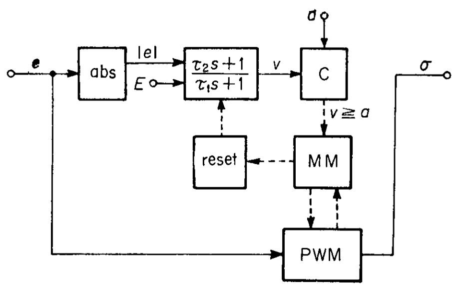

Fig. 12 Rate sensitive neural PFM-PWM.

3.1 Description of R.S. NPFM-PWM Modulator

The modulator of R. S. NPFM-PWM is shown in Fig. 12 and is defined as follows:

$$\sigma(t)=\sum_i \operatorname{sgn}[e(t_i)]h[\mathrm{H}(t-t_i)-\mathrm{H}(t-t_i-w_i)] \tag{24}$$

$$\left.\begin{aligned} &t_i=\min[t:\ t\geq t_{i-1}+T_m,\ v(t)\geq a] \\ &v(t)=\alpha(|e(t_i)|+E)+\frac{1-\alpha}{\tau_1}\int_{t_{i-1}}^{t}\exp\left(\frac{\tau-t}{\tau_1}\right)(|e(\tau)|+E)d\tau \end{aligned}\right\} \tag{25}$$

$$w_i = \min\,[k|e(t_i)|,\ w_M] \tag{26}$$

where e: modulator input; σ: modulator output; τ_1, τ_2: time constants ($\tau_2/\tau_1=\alpha<1$); a: threshold; E: positive bias ($<a$); h: pulse height; T_m: minimum pulse interval; w_M: maximum pulse width ($\leq T_m$) and k: positive constant.
(For detailed description, see part (I) of this series.)

3.2 Analysis of Oscillations

Let us consider the oscillations in the feedback system shown in Fig. 11, in the case that the transfer function of the linear system is

$$G(s)=\frac{C}{s}G_1(s)$$

where C is the gain constant of the integrator and $G_1(s)$ is a linear subsystem.

For the symmetric self-excited oscillation in the feedback system, the square wave approximation shown in Fig. 13 holds as developed in 2.2.1., if we replace $1/w$ with h and w with kA'. Thus, we have the quasi-describing function

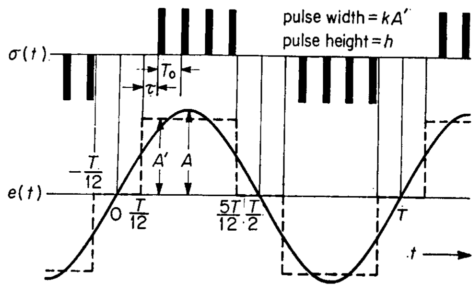

Fig. 13 Response of the PFWM to an approximated sinusoidal wave (1)

$$G_{QD_1}=\frac{4h}{\pi A}\sin\frac{\omega w}{2}\,\frac{\sin\frac{N\omega T_0}{2}}{\frac{\omega T_0}{2}}\exp\left[j\left\{\frac{\pi}{3}-\omega\left(\tau+\frac{N-1}{2}T_0+\frac{w}{2}\right)\right\}\right] \tag{28}$$

where

$$A=\frac{2\sqrt{3}}{\pi}A', \qquad \tau=\tau_1\ln\frac{(1-\alpha)A'}{A'+E-a},$$

$$T_0=\tau_1\ln\frac{(A'+E)(1-\alpha)}{A'+E-a}, \qquad w=kA'$$

As stated before, the point of intersection of loci of $-1/G_{QD_1}$ and $G(j\omega)$ gives the amplitude A and angular frequency ω of the oscillation and pulse number N per half period.

Examples of self-excited oscillations in the feedback system calculated by analog computer are shown in Figs. 14 and 15. In these examples, $a=1$, $E=0.9$, $\tau_1=1$, $\alpha=0.05$, $T_m=w_M=$

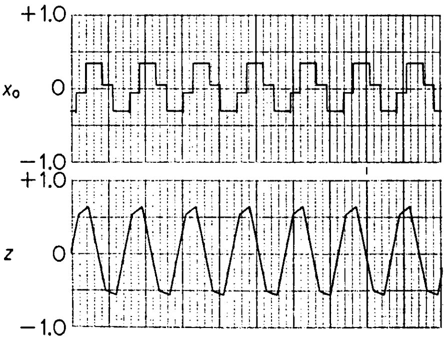

Fig. 14 4-pulse type self-excited oscillation.

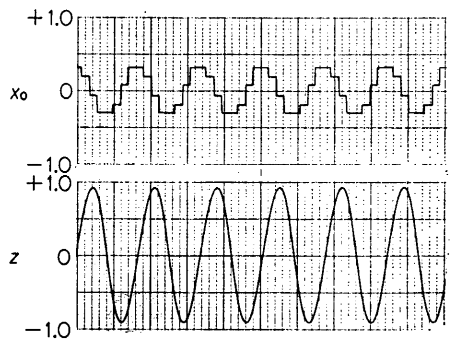

Fig. 15 6-pulse type self-excited oscillation.

Table 5 Symmetric Oscillation.
$\left(G(s)=\frac{C}{s}\,\frac{K_1}{T_1s+1}\right)$

	G_{QD_1} method		analog computer	
$2N$	A	ω	A	ω
2	0.15	1	0.23	0.84
	0.16	1	0.24	1.14
	0.22	1	0.56	0.90
4	0.35	1	0.45	0.92
	0.38	1	0.48	0.94
	0.71	1	0.66	0.76
6	0.67	1	1.05	0.89
	0.66	1	1.00	0.95
	0.74	1	1.10	0.99

Table 6 Symmetric Oscillation.
$\left(G(s)=\frac{C}{s}\,\frac{K_2}{s^2+Fs+K_2}\right)$

	G_{QD_1} method		analog computer	
$2N$	A	ω	A	ω
4	0.50	1	0.50	1.05
	0.58	1	0.55	1.08
	1.08	1	1.17	1.07
	1.28	1	1.21	1.08
6	0.90	1	0.85	1.05
	1.02	1	0.92	1.07

0.245, $k=0.0815$, $h=5$ and $G(s)=2/s\cdot 45/(23.3s+1)$ in Fig. 14, $G(s)=1.9/s\cdot 0.846/(s^2+0.714s+0.846)$ in Fig. 15. Table 5 and 6 show the results of analysis of oscillations in the feedback system with different parameters where $G(s)=C/s\cdot K_1/(T_1s+1)$ and $G(s)=C/s\cdot K_2/(s^2+Fs+K)$. Results by G_{QD_1}method in Table 5 are not so good as those in Table 6 for the reason that waveforms of oscillations in Table 6 can be considered to be sinusoidal while those in Table 5 cannot, as is seen in Figs. 14 and 15.

The trapezoidal wave approximation is not applicable to the combined modulator. Furthermore, G_{QD_3} method is not applicable to the modulator, for a_0 cannot be estimated for this case.

4. Conclusion

Self-excited oscillations in R. S. NPFM feedback systems and in R. S. NPFM-PWM feedback systems are analyzed using specially defined three kinds of quasi-describing function methods for the former and one for the latter. The methods will be shown to be applicable to NPFM feedback systems.

Examples with their experimental verification are introduced which clarify justification of the use and importance of the quasi-describing function methods.

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