

A Generalization of a Result of C.-C. Yang

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Yang [1] determined two entire functions of hyper order less than 1 which together with their first derivatives have the same zeros with the same multiplicities. Following his argument, we shall generalize his result.

Theorem *Let $f(z)$ and $g(z)$ be non-constant entire functions and let $g(z)$ have at most simple zeros. Further we assume that $\alpha(z)$, $\beta(z)$ and $\xi(z)$ are non-constant entire functions of order less than 1. Then there exist entire solutions $f(z)$ and $g(z)$ (as above) satisfying*

$$(1) \quad \frac{f(z)}{g(z)} = e^{\alpha(z)} \quad \text{and} \quad (2) \quad \frac{f'(z)}{g'(z)} = \xi(z) e^{\beta(z)}$$

if and only if $\beta(z) - \alpha(z) \equiv c'$ and that, when we put $c = e^{c'}$, either $c\xi'(z) - \alpha'(z) \equiv \text{constant}$ and $c\xi(z) - 1$ has only simple zeros, or there exists an entire function $\eta(z) \not\equiv 0$ with at most simple zeros such that

$$\frac{\eta'(z)}{\eta(z)} = \frac{c\xi(z)\alpha'(z)}{c\xi(z) - 1}$$

is an entire function.

<Proof> Assume that non-constant entire functions f and g , where g has at most simple zeros, satisfy the relations (1) and (2). Then we have

$$f'(z) = g'(z)e^{\alpha(z)} + \alpha'(z)g(z)e^{\alpha(z)} = \xi(z)g'(z)e^{\beta(z)}$$

Hence we have

$$(3) \quad g'(z) \{ \xi(z)e^{\beta(z)} - e^{\alpha(z)} \} = \alpha'(z)g(z)e^{\alpha(z)}.$$

By the assumption that $g(z)$ has only simple zeros, g and g' have no common zeros. Hence the set of the zeros of $g'(z)$ are contained to the set of the zeros of $\alpha'(z)$. Thus we have

$$(4) \quad \alpha'(z) = g'(z)h(z)$$

for some entire function $h(z)$. Substituting this in (3), we have

$$(5) \quad g(z)h(z) = \xi(z)e^{\beta(z) - \alpha(z)} - 1$$

Differentiating this and using (4), we obtain

$$(6) \quad g(z)h'(z) = \{\xi'(z) + \xi(z)(\beta'(z) - \alpha'(z))\} e^{\beta(z) - \alpha(z)} - \alpha'(z).$$

From (5), (4) and (6), we deduce

$$\begin{aligned} g'(z)h'(z) \{\xi(z)e^{\beta(z) - \alpha(z)} - 1\} &= g'(z)h'(z)g(z)h(z) = g(z)h'(z)\alpha'(z) \\ &= \alpha'(z) [\{\xi'(z) + \xi(z)(\beta'(z) - \alpha'(z))\} e^{\beta(z) - \alpha(z)} - \alpha'(z)] \end{aligned}$$

Hence we have

$$(7) \quad g'(z)h'(z) = \frac{\alpha'(z) [\{\xi'(z) + \xi(z)(\beta'(z) - \alpha'(z))\} e^{\beta(z) - \alpha(z)} - \alpha'(z)]}{\xi(z)e^{\beta(z) - \alpha(z)} - 1}$$

Multiplying $\xi(z)$ to both side of (7), we have

$$(7') \quad \xi(z)g'(z)h'(z) = \frac{\alpha'(z) [\{\xi'(z) + \xi(z)(\beta'(z) - \alpha'(z))\} \xi(z)e^{\beta(z) - \alpha(z)} - \xi(z)\alpha'(z)]}{\xi(z)e^{\beta(z) - \alpha(z)} - 1}$$

Here the left hand side of (7') is an entire function, whence the right hand side of (7') must be also an entire function. Assume here that $\beta(z) - \alpha(z) \equiv \text{constant}$. Then it can occur only when $\alpha'(z) \equiv 0$ or

$$(8) \quad \xi'(z) + \xi(z)(\beta'(z) - \alpha'(z)) - \xi(z)\alpha'(z) \equiv 0,$$

since the zeros of $\xi(z)e^{\beta(z) - \alpha(z)} - 1$ have the exponent of convergence not less than 1, and α , β and ξ are of order less than 1. Because of (3), $\alpha'(z) \equiv 0$ implies that $g(z)$ is constant, which contradicts to the assumption that g is not constant. But the relation (8) implies

$$(9) \quad \frac{\xi'(z)}{\xi(z)} = 2\alpha'(z) - \beta'(z).$$

Now the right hand side of (9) is an entire function, while the left hand side of it has poles, since $\xi(z)$ has zeros. ($\xi(z)$ is a non-constant entire function of order less than 1.) This leads to a contradiction. Thus we have proved $\beta(z) - \alpha(z) \equiv \text{const.} = c'$. Put $c = e^{c'}$. Then from (5), we have

$$(10) \quad g(z)h(z) = c\xi(z) - 1,$$

and from (6),

$$(11) \quad g(z)h'(z) = c\xi'(z) - \alpha'(z).$$

From (10) and (11), we have

$$(12) \quad \frac{h'(z)}{h(z)} = \frac{c\xi'(z) - \alpha'(z)}{c\xi(z) - 1}$$

If $c\xi'(z) \equiv \alpha'(z)$, we have by (11) $h(z) \equiv \text{constant} = A \neq 0$, whence by (10) $g(z) = A^{-1}(c\xi(z) - 1)$. Thus $c\xi(z) - 1$ has only simple zeros.

If $c\xi' \not\equiv \alpha'$, then from (10) and (12), we have

$$(13) \quad \eta(z)h(z) = (c\xi(z) - 1)e^{\alpha(z) + \gamma(z)}$$

$$(14) \quad \eta(z)h'(z) = (c\xi'(z) - \alpha'(z))e^{\alpha(z)+r(z)}$$

for some entire functions $\eta(z) \not\equiv 0$ with at most simple zeros and $\gamma(z)$. Differentiating (13),

$$h'(z) = \frac{[\{c\xi'(z) + (c\xi(z) - 1)(\alpha'(z) + \gamma'(z))\}\eta(z) - \eta'(z)(c\xi(z) - 1)]e^{\alpha(z)+r(z)}}{(\eta(z))^2}$$

Hence

$$\eta(z)h'(z) = [\{c\xi'(z) + (c\xi(z) - 1)(\alpha'(z) + \gamma'(z))\} - \frac{\eta'(z)}{\eta(z)}(c\xi(z) - 1)]e^{\alpha(z)+r(z)}$$

Noting (14), we deduce

$$(15) \quad \gamma'(z) = \frac{\eta'(z)}{\eta(z)} - \frac{c\xi(z)\alpha'(z)}{c\xi(z) - 1},$$

which is necessarily entire, since $\gamma(z)$ is entire.

Conversely, assume that $c\xi'(z) \equiv \alpha'(z)$ and $c\xi(z) - 1$ has only simple zeros. Then by (11) we have $h(z) \equiv \text{constant} = A \neq 0$. Noting (10), the functions

$$f(z) = A(c\xi(z) - 1)e^{\alpha(z)} \text{ and } g(z) = A(c\xi(z) - 1)$$

satisfy the relations (1) and (2).

Next we assume that there exists an entire function $\gamma(z)$ satisfying the relation (15), where $\eta(z)$ is an entire function with at most simple zeros. Then it is reasonable to set, from (10) and (13),

$$f(z) = \eta(z)e^{-r(z)} \text{ and } g(z) = \eta(z)e^{-\alpha(z)-r(z)}$$

Then f and g satisfy the relations (1) and (2). In fact, as

$$f'(z) = (\eta'(z) - \eta(z)\gamma'(z))e^{-r(z)} \text{ and } g'(z) = \{\eta'(z) - \eta(z)(\alpha'(z) + \gamma'(z))\}e^{-\alpha(z)-r(z)},$$

using (15), we have

$$\frac{f'(z)}{g'(z)} = c\xi(z)e^{\alpha(z)} = \xi(z)e^{\beta(z)},$$

which is to be proved.

References

- [1] C.-C. Yang: On two entire functions which together with their first derivatives have the same zeros.
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- [2] E. C. Titchmarsh: The theory of functions. 2nd ed. Oxford Univ. Press.