

Note on a Theorem of Erdős and Rényi

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1. Let $f(z)$ be regular near $z=0$, and denote by r_n the absolute value of the zero z_n of $f^{(n)}(z)$ which is nearest to the origin. Here $f^{(n)}(z)$ is the n -th derivative of $f(z)$. Then one of the results of Erdős and Rényi ([2] p. 140) is the following;

Theorem A. Let $f(z)$ be an entire function and $M(r) = \max_{|z|=r} |f(z)|$. Then by denoting $x=H(y)$ the inverse function of $y=\log M(x)$, we have

$$(1) \quad \lim_{n \rightarrow \infty} \frac{H(n)}{nr_n} \leq \frac{e}{\log 2}$$

We shall give a slight generalization of this result. Denote by ρ_n the radius of univalence of $f^{(n)}(z)$. That is, ρ_n is the supremum of r such that for $|z| < r$, $f^{(n)}(z)$ is univalent. Then our assertion is as follows;

Theorem Let $f(z)$ be an entire function of finite lower order λ and $M(r) = \max_{|z|=r} |f(z)|$. Then denoting by $x=H(y)$ the inverse function of $y=\log M(x)$, we have

$$(2) \quad \lim_{n \rightarrow \infty} \frac{H(n)}{n\rho_{n-1}} \leq \frac{e^{1-\lambda}}{\log 2}$$

Here lower order λ is defined by $\lambda = \lim_{r \rightarrow \infty} \left(\frac{\log \log M(r)}{\log r} \right)$. For the proof of our result, we shall follow the method of Erdős and Rényi and use, in addition, the strong lemma due to Shah and Trimble.

Let denote by $\mu(r)$ the maximum term of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for $|z|=r$ and by $\nu(r)$ its central index:

$$\mu(r) = \max_{k \geq 0} |a_k| r^k, \quad \nu(r) = \max \{n; \mu(r) = |a_n| r^n\}$$

Lemma 1. ([6]) If $f(z)$ is an entire function of lower order λ , then

$$(3) \quad \lim_{r \rightarrow \infty} \frac{\nu(r)}{\log \mu(r)} \leq \lambda$$

Lemma 2. ([7]) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function and $\nu(r)$ its central index. For $n=1, 2, \dots$, let $\chi_n = n(1-2^{-1/n+2})$. Then for $r \geq r_0$, we have

$$(4) \quad r \cdot \chi(\nu(r)-1) < (\nu(r)-1) \cdot \rho(\nu(r)-1)$$

2. Proof of Theorem. We may assume $|f(0)|=1$. In this case (see for example [8] p. 31) we have

$$(5) \quad \log \mu(r) = \int_0^r \frac{\nu(t)}{t} dt$$

It follows from (5) that if $c > 1$, taking into account that $\nu(t)$ is non-decreasing, we have

$$(6) \quad \log \mu(cr) - \log \mu(r) = \int_r^{cr} \frac{\nu(t)}{t} dt \geq \nu(r) \log c$$

By the assumption that $f(z)$ is an entire function of finite lower order λ , we have by lemma 1,

$$(7) \quad \lim_{r \rightarrow \infty} \frac{\nu(r)}{\log \mu(r)} \leq \lambda$$

Thus for any $\epsilon > 0$, there can be found a sequence $r_n (n=1, 2, \dots)$ for which $r_n \rightarrow \infty$ and $\nu(r_n) \leq (\lambda + \epsilon) \log \mu(r_n)$. Applying (6) for $r=r_n$, we obtain

$$(8) \quad \nu(r_n) \left(\log c + \frac{1}{\lambda + \epsilon} \right) \leq \log \mu(cr_n)$$

Choosing $c = e^{1-\lambda+\epsilon}$, it follows that

$$(9) \quad \nu(r_n) \leq \log \mu(r_n e^{1-\lambda+\epsilon}) \leq \log M(r_n e^{1-\lambda+\epsilon})$$

Hence we have

$$(10) \quad H(\nu(r_n)) \leq r_n e^{1-\lambda+\epsilon}$$

Now apply lemma 2 to (10) as $r=r_n$ ($n \geq n_0$), then we have

$$(11) \quad \frac{H(\nu(r_n))}{\nu(r_n) \cdot \rho(\nu(r_n)-1)} \leq \frac{e^{1-\lambda+\epsilon}}{\chi(\nu(r_n)-1)}$$

Taking into account $\chi(n) \rightarrow \log 2$ as $n \rightarrow \infty$, we obtain

$$(12) \quad \lim_{n \rightarrow \infty} \frac{H(n)}{n \rho(n-1)} \leq \frac{e^{1-\lambda+\epsilon}}{\log 2}$$

As (12) is valid for any $\epsilon > 0$, the assertion of theorem is proved.

Corollary Under the same assumption of theorem, we have

$$(13) \quad \lim_{n \rightarrow \infty} \frac{H(n)}{n \cdot r_n} \leq \frac{e^{1-\lambda}}{\log 2}$$

References

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