

Notes on the Lusternik-Schnirelmann Category and H-spaces

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1. It is well known that if X is an H-space, then ΩX is a homotopy commutative H-space, and for this converse there exists Sugawara's Theorem (Mem. Coll. Sci. Univ. Kyoto, Ser. A 33 (1960) Th. 8.1.): i.e. $\pi_i(B)=0$ for $i \leq n$, $i > 3n+1$ ($n \geq 0$), and ΩB is a homotopy-commutative H-space, then B is an H-space.

And that, it is well known that S^1 , S^3 and S^7 are only H-spaces for spheres, then only S^1 is fulfilled for the condition of Sugawara's Theorem ($n=0$), but S^3 and S^7 is not contained in this theorem.

We show that if $\text{cat } X \leq 2$ ($\text{cat } X$ means Lusternik-Schnirelmann Category of X) and ΩX is a homotopy-commutative H-space, then X is an H-space. This theorem contain the spaces S^3 and S^7 , but this theorem is not best possible, because it is well known that RP^i (i -dim. real projective spaces) ($i=3, 7$.) is H-space, but it is easy to see that $\text{cat } RP^i = i+1$, i.e. $\text{cat } RP^i > 2$ ($i=3, 7$).

And that, we easily obtain from the result for T. Ganea [2], if n is odd integers ≥ 1 , then $\text{cocat } S^n \leq 3$ and $\text{cocat } S^n = 2$ if and only if $n=1, 3, 7$. This proof is quite analogous of proof of Prop. 1. 2 for T. Ganea [2].

2. All spaces will be assumed to have base points and to have the homotopy type of connected CW-complexes. All maps and homotopies will respect to base points.

The category and cocategory, $\text{cat } X$ and $\text{cocat } X$, of X are the strictly positive integers given by the following inductive ;

DEFINITION 1.

$\text{cat } X = 1$ if and only if X is contractible ; $\text{cat } X \leq n+1$ whenever there exists a cofibration $P \rightarrow Y \rightarrow A$ such that the cofibre A dominates X and $\text{cat } Y \leq n$. If $\text{cocat } X \neq n$ for all $n \geq 1$, we put $\text{cat } X = \infty$.

DEFINITION 2.

$\text{cocat } X = 1$ if and only if X is contractible ; $\text{cocat } X \leq n+1$ whenever there exists a fibration $Q \rightarrow Y \rightarrow B$ such that the fibre Q dominates X and $\text{cocat } Y \leq n$. If $\text{cocat } X \neq n$ for all $n \geq 1$, we put $\text{cocat } X = \infty$.

$X \neq n$ for all $n \geq 1$, we put $\text{cocat } X = \infty$.

As an immediate consequence of this definition we have that the category and cocategory are based homotopy type invariants.

PROPOSITION 1. (T. Ganea [1] Th. 2. 12.)

$$\text{nil } \Omega X \leq \text{cocat } X - 1.$$

It is well known that

1) $\text{cat } X \leq 2$ if and only if X is an H' -space,

2) $\text{cocat } X \leq 2$ if and only if X is an H -space.

By T. Ganea [1] Th. 2.5, Cor. 2. 6 ; $\text{cocat } X \leq 2$ if and only if X is dominated by $\Omega \Sigma X$. We show that this theorem is dualized as follows.

LEMMA 2.

$\text{cat } X \leq 2$ if and only if X is dominated by a suspension space.

PROOF.

Suppose X is dominated by a ΣA , by def.2 trivially we have $\text{cat } X \leq 2$. Conversely, suppose $\text{cat } X \leq 2$ and let $P \rightarrow Y \rightarrow A$ be a cofibration such that A dominates X and $\text{cat } Y = 1$; then Y is contractible, and this may be shown to imply that A has the homotopy type of ΣP .

COROLLARY 3.

$\text{cat } X \leq 2$ if and only if X is dominated by $\Sigma \Omega X$.

PROOF.

For any space Z , let the maps

$$e : Z \rightarrow \Omega \Sigma Z \text{ and } p : \Sigma \Omega Z \rightarrow Z$$

be defined by $e(z)(s) = \langle z, s \rangle$ and $p \langle w, s \rangle = w(s)$. Suppose $\text{cat } X \leq 2$. Then, by lemma 1, there is a space A and maps

$$X \xrightarrow{\varphi} \Sigma A \xrightarrow{\psi} X$$

such that $\psi \cdot \varphi$ is homotopic to the identity map of X . With $\Sigma \Omega \psi$ and Σe defined in the obvious way, the composition

$$X \xrightarrow{\varphi} \Sigma A \xrightarrow{\Sigma e} \Sigma \Omega \Sigma A \xrightarrow{\Sigma \Omega \psi} \Sigma \Omega X \xrightarrow{p} X$$

is easily seen to equal $\psi \cdot \varphi$ so that $\Sigma \Omega X$ dominates X . The converse follows directly from lemma 1.

PROPOSITION 4. (J. Stasheff [4] Th. 1. 10.)

There exists a map $f : \Sigma \Omega X \times \Sigma \Omega X \rightarrow X$ which is axial with respect to p (in cor. 3) if and only if ΩX is a homotopy-commutative.

A map f is axial with respect to p if $f|_{\Sigma\Omega X \vee \Sigma\Omega X} = p$.

THEOREM 5.

If $\text{nil } \Omega X = 1$ (i.e. ΩX is a homotopy-commutative) and $\text{cat } X \leq 2$, then $\text{cocat } X \leq 2$.

PROOF.

By cor.3, X is dominated by $\Sigma\Omega X$; i.e. $p \cdot q \simeq 1_x$ ($q = \Sigma\Omega\phi \cdot \Sigma e \cdot \phi$), and by prop.4, there exists a map f which is axial with respect to p . We define a multiplication on X as follows,

$$h(x, y) = f(q(x), q(y)) \text{ for all } x, y \in X.$$

Then X is an H-space with a multiplication h , i.e. $\text{cocat } X \leq 2$.

In th.5, if X has a same homotopy type of suspension space, then naturally $\text{cat } X \leq 2$. Hence if $\text{nil } \Omega X = 1$, then $\text{cocat } X \leq 2$. This is more generalized in M. Arkowitz (Pacific J. Math. vol. 12 (1962)); for ι is a identity class of $\Pi(\Sigma A, \Sigma A)$, if $[\iota, \iota] = 0$, then $\text{cocat } \Sigma A \leq 2$. Indeed, if $\text{nil } \Omega \Sigma A = 1$, then it follows $[\iota, \iota] = 0$.

REMARK.

Let $X(n)$ result by killing off the homotopy groups of X in dimension $> n$. In I. Berstein, T. Ganea (Illinois J. of Math. 5, 112p), we define a space $Y = CP^2(5)$; then ΩY is a homotopy-commutative but Y fails to be an H-space. Indeed, since Y has the rational cohomology groups of CP^2 and $\text{cat } CP^2 = 3$, hence $\text{cat } Y > 2$.

By th.5, it follows that the special case ($\text{nil } \Omega X = 1$) of equality of prop.1 is proved if $\text{cat } X \leq 2$.

PROPOSITION 6. (T. Ganea [2] Prop. 1.2.)

$\text{nil } \Omega S^n \leq 2$ for any odd $n \geq 1$; $\text{nil } \Omega S^n = 1$ if and only if $n = 1, 3, 7$.

THEOREM 7.

$\text{cocat } S^n \leq 3$ for any odd $n \geq 1$; $\text{cocat } S^n = 2$ if and only if $n = 1, 3, 7$.

PROOF.

If n is odd, the infinite exact sequence of James [4],

$$\cdots \rightarrow \pi_r(S^n) \rightarrow \pi_r(\Omega \Sigma S^n) \xrightarrow{k_*} \pi_r(\Omega \Sigma S^{2n}) \rightarrow \pi_{r-1}(S^n) \rightarrow \cdots$$

reveals that S^n has the homotopy type of the fibre of the fibre map which is homotopically equivalent to $k: \Omega \Sigma S^n \rightarrow \Omega \Sigma S^{2n}$. Since a $\text{cocat } \Omega \Sigma S^n \leq 2$, one has $\text{cocat } S^n \leq 3$.

In th.7, we show that the special case of equality of prop.1 is proved if X is S^n for any odd $n \geq 1$.

COROLLARY 8.

W-long $S^n \leq 2$ for any odd integers $n \geq 1$.

i.e. for any $\alpha_i \in \pi_{q_i}(S^n)$ ($i=1, 2, 3$), $[\alpha_1, \alpha_2, \alpha_3]=0$ for all odd integers $n \geq 1$.

REFERENCES.

1. T. Ganea : Lusternik-Schnirelmann category and cocategory.
Proc. of London Math. Soc. vol. 10 (1960), 437-451 p.
2. : On the loop spaces of projective spaces.
J. Math. and Mech. vol. 16 (1967), 853-858 p.
3. I. M. James : On the suspension triad of a sphere.
Ann. of Math. vol. 63 (1956), 407-429 p.
4. J. Stasheff : On homotopy abelian H-spaces.
Proc. Cambridge Phil. Soc. vol 57. (1961), 734-745 p.