

Notes on Pringsheim's Theorem and Hadamard's gap-theorem II

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1. Introduction

In earlier notes [3], by an application of the Hadamard singularity criterion we obtained the following theorems.

Theorem 1. Suppose that the series $\sum c_n z^n$ has radius of convergence 1. For suitable positive numbers δ', δ ($\delta' > \pi^{\frac{1}{2}} \delta > 0$) and a real number p , if $\operatorname{Re} c_n \geq -\delta n^p$ and there exists an infinite subsequence $\{n_k\}$ such that $\operatorname{Re} c_{n_k} \geq \delta' n_k^{p+\frac{1}{2}}$, then $z=1$ is a singular point of the function defined by the series $\sum c_n z^n$.

Theorem 2. Suppose that the series $\sum c_n z^n$ has radius of convergence 1, and let $\{n_k\}$ be an increasing sequence of positive integers such that $n_{k+1} - n_k \geq \alpha n_k$ for $k=0, 1, 2, \dots$ where $\alpha > 0$. For suitable positive numbers δ', δ ($\delta' > \pi^{\frac{1}{2}} \delta > 0$) and a real number p , if $|c_{n_k}| \geq \delta' n_k^{p+\frac{1}{2}}$ and $|c_n| \leq \delta n^p$ for $n \neq n_k$ $k=0, 1, 2, \dots$, then the unit circle is a natural boundary of the function defined by the series $\sum c_n z^n$.

The purpose of the present paper is to prove the following results.

If in these theorems $\{n_k\}$ contains an infinite number of multiples of s (a positive integer), we can replace the condition $\delta' > \pi^{\frac{1}{2}} \delta > 0$ with the condition $\delta' > \left(\frac{2\pi}{s+1}\right)^{\frac{1}{2}} \delta > 0$.

Theorem 1. Theorem 2. are obtained as special cases $s=1$.

For the proofs of the theorems we need several Lemmas. In this paper we may put

$$\left| \sum_{m=0}^n \binom{n}{m} h^m c_m \right| \stackrel{1}{=} \varphi_n(h, c)$$

2. Lemmas

Lemma 1. [5] Suppose that the series $\sum c_n z^n$ has radius of convergence 1. A necessary and sufficient condition that $z=1$ be a singular point of the function defined by the series $\sum c_n z^n$ is that

$$(1) \quad \limsup_{n \rightarrow \infty} \varphi_n(h, c) = 1 + h$$

for some $h > 0$.

Lemma 2. Suppose that the series $\sum c_n z^n$ has radius of convergence 1. A necessary and sufficient condition that $|z|=1$ be a natural boundary of the function defined by the series $\sum c_n z^n$ is that

$$(2) \quad \limsup_{n \rightarrow \infty} \varphi_n(h e^{i\theta}, c) = 1 + h$$

for each θ ($0 \leq \theta < 2\pi$)

Proof. Replacing z of $f(z)$ by $e^{i\theta}z$, $z = e^{i\theta}$ is a singular point if and only if

$$(3) \quad \limsup_{n \rightarrow \infty} \varphi_n(h e^{i\theta}, c) = 1 + h$$

Lemma 3. If the series $\sum c_n z^n$ has radius of convergence 1,

$$(4) \quad \limsup_{n \rightarrow \infty} \varphi_n(h e^{i\theta}, c) \leq 1 + h$$

for each θ ($0 \leq \theta < 2\pi$).

Lemma 4. When s is an arbitrary positive integer and n is sufficiently large, there exists r (< 1) which the following inequalities hold for $0 < \theta < s^{-1}$.

$$(5) \quad \left(\begin{matrix} (s+1)n \\ [(1+\theta)sn] \end{matrix} \right) s^{[(1+\theta)sn]} < r^n (s+1)^{(s+1)n}$$

$$(6) \quad \left(\begin{matrix} (s+1)n \\ [(1-\theta)sn] \end{matrix} \right) s^{[(1-\theta)sn]} < r^n (s+1)^{(s+1)n}$$

where $[\]$ is the Gaussian symbol and r is independent on n .

Proof. By Stirling's formula $n! = (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n+\frac{\lambda}{12n}}$ ($0 < \lambda < 1$)

$$(7) \quad \left(\begin{matrix} (s+1)n \\ [(1+\theta)sn] \end{matrix} \right) s^{[(1+\theta)sn]} < n \left\{ \frac{(1+\theta)^s}{(1-\theta)s} \right\}^{-n} (s+1)^{(s+1)n}$$

Since $(1+\theta)^{(1+\theta)s} (1-\theta)s^{(1-\theta)s} > 1$, the inequality (5) holds. By similar discussion, we have (6).

Lemma 5. When p is an arbitrary real number and n is sufficiently large, we have the following inequality for an arbitrary number A (> 1).

$$(8) \quad \sum_{m=0}^{(s+1)n} \binom{(s+1)n}{m} s^m m^p < A (sn)^p (s+1)^{(s+1)n}$$

Proof. From **Lemma 4**.

(i) When $p > 0$, For a number θ such that $0 < \theta < \left(\frac{A+1}{2}\right)^{\frac{1}{p}} - 1$,

$$(9) \quad \sum_{m=0}^{[(1-\theta)sn]} \binom{(1+s)n}{m} s^m m^p < r^n (s+1)^{(s+1)n} [(1-\theta)sn]^p \{[(1-\theta)sn] + 1\} < r_1^n (s+1)^{(s+1)n} \quad (0 < r < r_1 < 1),$$

where r, r_1 are independent on n .

By similar discussion

$$(10) \quad \sum_{m=[(1+\theta)sn]}^{(s+1)n} \binom{(s+1)n}{m} s^m m^p < r_2^n (s+1)^{(s+1)n},$$

where r_2 (< 1) is independent on n .

$$(11) \quad \sum_{m=[(1-\theta)sn]+1}^{[(1+\theta)sn]-1} \binom{(s+1)n}{m} s^m m^p < \sum_{m=[(1-\theta)sn]+1}^{[(1+\theta)sn]-1} \binom{(s+1)n}{m} s^m \{[(1+\theta)sn] - 1\}^p < (1+\theta)^p s^p n^p (s+1)^{(s+1)n}.$$

From (9) + (10) + (11)

$$(12) \quad \sum_{m=0}^{(s+1)n} \binom{(s+1)n}{m} s^m m^p < \{(r_1^n + r_2^n) s^{-p} n^{-p} + (1+\theta)^p\} s^p n^p (s+1)^{(s+1)n} \\ < A(s n)^p (s+1)^{(s+1)n},$$

because we may take n such that $(r_1^n + r_2^n) s^{-p} n^{-p} < \frac{A-1}{2}$.

(ii) When $p < 0$, by similar discussion in (i), for a number θ such that

$$0 < \theta < 1 - \left(\frac{2}{A+1}\right)^{-\frac{1}{p}},$$

$$(13) \quad \sum_{m=0}^{\lfloor (1-\theta)sn \rfloor} \binom{(s+1)n}{m} s^m m^p < r_3^n (s+1)^{(s+1)n}.$$

$$(14) \quad \sum_{m=\lfloor (1+\theta)sn \rfloor}^{(s+1)n} \binom{(s+1)n}{m} s^m m^p < r_4^n (s+1)^{(s+1)n}.$$

$$(15) \quad \sum_{m=\lfloor (1-\theta)sn \rfloor+1}^{\lfloor (1+\theta)sn \rfloor-1} \binom{(s+1)n}{m} s^m m^p < (1-\theta)^p s^p n^p (s+1)^{(s+1)n}.$$

From (13) + (14) + (15)

$$(16) \quad \sum_{m=0}^{(s+1)n} \binom{(s+1)n}{m} s^m m^p < A(s n)^p (s+1)^{(s+1)n}.$$

(iii) When $p=0$ it is evident. q. e. d.

Lemma 6. For an arbitrary positive number ε and sufficiently large n

$$(17) \quad \binom{(s+1)n}{sn} s^{sn} > (1-\varepsilon) \left(\frac{s+1}{2\pi}\right)^{\frac{1}{2}} (sn)^{-\frac{1}{2}} (s+1)^{(s+1)n}$$

Proof. Again, by Stirling's formula it is evident.

3. A note on Pringsheim's theorem

Theorem 3. Suppose that the series $\sum c_n z^n$ has radius of convergence **1**. For suitable a real number p , a positive integer s and positive numbers δ', δ ($\delta' > \left(\frac{2\pi}{s+1}\right)^{\frac{1}{2}}$, $\delta > 0$), if $\operatorname{Re} c_n \geq -\delta n^p$ and there exists an infinite subsequence $\{n_k\}_{k=0,1,2,\dots}$, $n_k \equiv 0 \pmod{s}$, such that $\operatorname{Re} c_{n_k} \geq \delta' n_k^{p+\frac{1}{2}}$, then $z=1$ is a singular point of the function defined by the series $\sum c_n z^n$.

Proof. Putting $\operatorname{Re} c_n = \alpha_n$, it is clear that $\sum \alpha_n z^n$ has radius of convergence **1**. For $n_k = sN_k$

$$(18) \quad \sum_{m=0}^{(s+1)N_k} \binom{(s+1)N_k}{m} s^m \alpha_m \geq \binom{(s+1)N_k}{sN_k} s^{sN_k} \alpha_{n_k} + \sum_{m=0}^{*} \binom{(s+1)N_k}{m} s^m \alpha_m,$$

where the sign $*$ means that we drop non-negative terms.

For sufficiently large k , by **Lemma 5**. and putting

$$\left(\frac{s+1}{2\pi}\right)^{\frac{1}{2}} \delta' - \delta = 3\beta > 0, \quad A = \frac{\delta + 2\beta}{\delta + \beta} > 1,$$

$$(19) \quad \left| \sum_{m=0}^{*} \binom{(s+1)N_k}{m} s^m \alpha_m \right| < \frac{\delta(\delta + 2\beta)}{\delta + \beta} s^p n_k^p (s+1)^{(s+1)N_k}.$$

By **Lemma 6**. and putting $\varepsilon = \frac{\beta}{\delta + \beta}$

$$(20) \quad \binom{(s+1)N_k}{sN_k} s^{sN_k} \alpha_{n_k} > \frac{\delta(\delta + 3\beta)}{\delta + \beta} s^p n_k^p (s+1)^{(s+1)N_k}$$

From (18), (19) and (20)

$$(21) \quad \left| \sum_{m=0}^{(s+1)N_k} \binom{(s+1)N_k}{m} s^m \alpha_m \right| > \frac{\delta\beta}{\delta+\beta} s^{pN_k} (s+1)^{(s+1)N_k}$$

Accordingly we have

$$(22) \quad \limsup_{n \rightarrow \infty} \varphi_n(s, \alpha) \geq s+1$$

By **Lemma 3**.

$$(23) \quad \limsup_{n \rightarrow \infty} \varphi_n(s, c) = s+1$$

By **Lemma 1**, $z=1$ is a singular point. q. e. d.

Note 1. When $p \rightarrow -\infty$, we have Pringsheim's theorem.

4. A note on Hadamard's gap-theorem

Theorem 4. Suppose that the series $\sum c_n z^n$ has radius of convergence 1. For suitable a real number p , a positive integer s and positive numbers δ', δ ($\delta' > \left(\frac{2\pi}{s+1}\right)^{\frac{1}{2}} \delta > 0$), if there exists an increasing sequence of positive integers $\{n_k\}$, $n_k \equiv 0 \pmod{s}$, such that

$$|c_{n_k}| \geq \delta' n_k^{p+\frac{1}{2}}, \quad |c_n| \leq \delta n^p$$

for $(1-\theta)n_k \leq n \leq (1+\theta)n_k$, $n \neq n_k$, with a fixed positive θ ($< s^{-1}$), then $|z|=1$ is a natural boundary of the function defined by the series $\sum c_n z^n$.

Proof. For sufficiently large k and each t ($0 \leq t < 2\pi$), by putting $n_k = sN_k$,

$$(24) \quad \left(\frac{s+1}{2\pi}\right)^{\frac{1}{2}} \delta' - \delta = 4\beta, \quad A = \frac{\delta+2\beta}{\delta+\beta} > 1 \quad \text{and} \quad \varepsilon = \frac{\beta}{\delta+\beta},$$

$$\left| \sum_{m=0}^{(s+1)N_k} \binom{(s+1)N_k}{m} s^m c_m e^{imt} \right| \geq \left(\binom{(s+1)N_k}{sN_k} s^{sN_k} |c_{n_k}| \right. \\ \left. - \sum_{m=0}^{(s+1)N_k} \binom{(s+1)N_k}{m} s^m |c_m| \right),$$

where the sign' indicates that the term corresponding to $m=n_k$ is omitted in the summation.

By similar discussion in the proof of Theorem 3.,

$$(25) \quad \left(\binom{(s+1)N_k}{sN_k} s^{sN_k} |c_{n_k}| \right) > \frac{\delta(\delta+4\beta)}{\delta+\beta} s^{pN_k} (s+1)^{(s+1)N_k}$$

$$(26) \quad \sum_{m=0}^{(s+1)N_k} \binom{(s+1)N_k}{m} s^m |c_m| < \sum_{\substack{[(1+\theta)sN_k]-1 \\ [(1-\theta)sN_k]+1}}^{[(1+\theta)sN_k]-1} \binom{(s+1)N_k}{m} s^m \cdot \delta m^p + r_3 N_k (s+1)^{(s+1)N_k} \\ < \frac{\delta(\delta+2\beta)}{\delta+\beta} s^{pN_k} (s+1)^{(s+1)N_k} + r_3 N_k (s+1)^{(s+1)N_k} \\ < \frac{\delta(\delta+3\beta)}{\delta+\beta} s^{pN_k} (s+1)^{(s+1)N_k},$$

where $0 < r_3 < 1$.

From (24), (25) and (26)

$$(27) \quad \left| \sum_{m=0}^{(s+1)N_k} \binom{(s+1)N_k}{m} s^m c_m e^{imt} \right| > \frac{\delta\beta}{\delta+\beta} s^{pN_k} (s+1)^{(s+1)N_k}$$

Accordingly we have

$$(28) \quad \limsup_{n \rightarrow \infty} \varphi_n(s e^{it}, c) \geq s+1$$

From **Lemma 3**.

$$(2) \limsup_{n \rightarrow \infty} \varphi_n(se^{it}, c) = s+1 \quad .$$

By **Lemma 2**. $|z|=1$ is a natural boundary. q. e. d.

Note 2. When $p \rightarrow -\infty$, we have the second Hadamard's gap-theorem.

Corollary 1. Let $f(z) = \sum c_n z^n$ has radius of convergence 1 and

$\{n_k\}$ be a sequence of positive integers satisfying Hadamard's gap condition,

i.e. $n_{k+1} - n_k \geq \alpha n_k$ for $k=0, 1, 2, \dots$, where $\alpha > 0$, and moreover $n_k \equiv 0 \pmod{s}$.

If $\delta' > \left(\frac{2\pi}{s+1}\right)^{\frac{1}{2}} \delta > 0$ and p is a real number such that

- (i) $|c_{n_k}| \geq \delta' n_k^{p+\frac{1}{2}} \quad (k=0, 1, 2, \dots)$
- (ii) $|c_n| \leq \delta n^p \quad (n \neq n_k \text{ for any } k),$

then $f(z)$ has the unit circle as a natural boundary.

Note 3. When $p \rightarrow -\infty$, we have the first Hadamard's gap-theorem. By similar discussion in this paper, Theorem 2. of [4] is extended as follows:

Theorem 5. Suppose that $\varphi(x)$ satisfies the conditions

- (a) $\lim_{x \rightarrow \infty} \frac{\varphi(x)}{x} = 0, \quad \lim_{x \rightarrow \infty} \frac{\varphi(x)}{\sqrt{x} \log x} = \infty \quad .$
- (b) $\varphi'(x) \downarrow 0 \quad \text{for } x \uparrow \infty,$

and that the series $\sum c_n z^n$ has radius of convergence 1, and let $\{n_k\}$ $k=0, 1, 2, \dots$ be an increasing sequence of positive integers such that $n_{k+1} - n_k \geq \varphi(n_k)$, then if there exists a subsequence $\{n_l\}$, $n_l \equiv 0 \pmod{s}$, of the sequence $\{n_k\}$, and $\{n_l\}$ satisfies the following conditions, then the unit circle is a natural boundary.

Conditions. For suitable positive numbers $\beta (1 > \beta > 0)$, δ', δ ($\delta' > \left(\frac{2\pi}{s+1}\right)^{\frac{1}{2}} \delta > 0$), a real number p and a positive number $\theta (< s^{-1})$, the following inequalities hold.

- (c) $\frac{e^{\frac{\varphi(n_l)^2}{n_l}}}{e^{n_l}} > |c_{n_l}| \geq \delta' n_l^{p+\frac{1}{2}}$
for $[(1-\theta)n_l] \leq n_l \leq [(1+\theta)n_l]$
- (d) $|c_n| \leq \delta n^p$
for $[(1-\theta)n_l] \leq n \leq [(1+\theta)n_l]$ and $n \notin \{n_k\}$.

References

- [1] Landau. E. Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. 2 Aufl., Springer, Berlin (1929)
- [2] Ilieff, L. Analytische Nichtfortsetzbarkeit und Überkonvergenz einiger Klassen von Potenzreihen, VEB Deutscher Verlag der Wissenschaften, Berlin (1960)
- [3] H. Sakashita Notes on Pringsheim's theorem and Hadamard's gap-theorem, Bull. Kyoto Gakuji Univ. Ser. B, No. 21 (1962)

- [4] ——— A condition for the non-continuability of power series. Ibid. No. 24 (1964)
- [5] J. P. King Tests for singular points. Amer. Math. Monthly. vol. 72 No. 8 (1965)