

The Theorem of Fatou-Pólya and Pólya-Hausdorff Space

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Introduction

In an earlier note [5] we obtained a constructive proof of the result of Fatou-Pólya that if $\sum a_n z^n$ has finite radius of convergence, a sequence $\{e_n\}$, taking the values ± 1 only, can be chosen such that $\sum a_n e_n z^n$ has its circle of convergence as a natural boundary.

The object of this paper is to give a constructive proof of an extended result of Fatou-Pólya and to show that this result contains the following result on Pólya-Hausdorff space [4]

The set of the continuable power series is the boundary of the set of the non-continuable power series.

§ 1

Theorem 1. Suppose that $\sum c_n z^n$ has radius of convergence 1 and $\{b_n\}$ is an arbitrary sequence, with $\limsup_{n \rightarrow \infty} |b_n|^{\frac{1}{n}} = 1$, then there exists a sequence $\{d_n\}$, where $d_n = b_n$ or 0, such that $\sum (c_n + d_n) z^n$ has the unit circle as a natural boundary.

Proof. Let $s_1, s_2, \dots$ be everywhere dense on $|z| = 1$ and $z_1, z_2, \dots$ be an infinite sequence in which each s_j occurs infinitely often.

Further, put

$$(1) \quad \varphi_n(c, z) = \frac{1}{2^{2n}} \sum_{\substack{m=0 \\ 3 \nmid m}}^{2n} \binom{2n}{m} c_m z^m$$

$$(2) \quad |\varphi_n(c, z)| = R_n(z), \quad \left| \frac{1}{2^{2n}} \binom{2n}{n} b_n \right| = r_n.$$

Next, let $\{N_k\}$, $\{n_k\}$ ($k=1, 2, \dots$) be two sequences of positive integers satisfying the following conditions.

$$(3) \quad \lim_{k \rightarrow \infty} |c_{N_k}|^{\frac{1}{N_k}} = 1, \quad \lim_{k \rightarrow \infty} |b_{n_k}|^{\frac{1}{n_k}} = 1.$$

$$(4) \quad 2n_k < n_{k+1}$$

$$(5) \quad n_k < N_k < n_{k+1},$$

Now we construct $\{d_n\}$ as follows:

$$(6) \quad d_n = \begin{cases} b_n & \text{for } n = n_k \text{ with } r_{n_k} > 2R_{n_k}(z_k) \\ 0 & \text{for the rest} \end{cases}$$

From (3) (5) and (6) $\sum (c_n + d_n)z^n$ has radius of convergence **1**.

From (1) (2) (4) and (6)

$$\begin{aligned} |\varphi_{n_k}(c+d, z_k)| &= \left| \frac{1}{2^{\frac{4}{3}n_k}} \sum_{\frac{2}{3}n_k}^{\frac{4}{3}n_k} \binom{2n_k}{m} (c_m + d_m) z_k^m \right| \\ &= \left| \frac{1}{2^{\frac{4}{3}n_k}} \left\{ \sum_{\frac{2}{3}n_k}^{\frac{4}{3}n_k} \binom{2n_k}{m} c_m z_k^m + \binom{2n_k}{n_k} d_{n_k} z_k^{n_k} \right\} \right| \geq \frac{1}{2} r_{n_k} \end{aligned}$$

By (2) and Stirling's formula

$$(7) \quad \lim_{k \rightarrow \infty} r_{n_k}^{\frac{1}{n_k}} = 1$$

Accordingly

$$(8) \quad \lim_{k \rightarrow \infty} |\varphi_{n_k}(c+d, z_k)|^{\frac{1}{n_k}} \geq 1$$

Since each s_j occurs infinitely often in $\{z_k\}$, we have

$$(9) \quad \limsup_{n \rightarrow \infty} |\varphi_n(c+d, s_j)|^{\frac{1}{n}} \geq 1$$

By the fact that $\{s_j\}$ is everywhere dense on $|z|=1$ and [3], $\sum (c_n + d_n)z^n$ has $|z|=1$ as a natural boundary. q. e. d.

Corollary 1. Suppose that $\sum c_n z^n$ has radius of convergence **1**, then a sequence $\{e_n\}$, taking the values ± 1 only, can be chosen such that $\sum a_n e_n z^n$ has the unit circle as a natural boundary (Fatou-Pólya).

Proof. In **Theorem 1** we have only to replace $\{b_n\}$ by $\{-2c_n\}$. q. e. d.

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the sets of the non-continuable and continuable power series respectively, then we have the following corollary.

Corollary 2. $\mathbf{S}_2$ is the boundary of $\mathbf{S}_1$.

Proof. $\{c_n\} \in \mathbf{S}_2$ and let $\{\varepsilon_n\}$ be a neighborhood of $\{c_n\}$. If we put $b_n = \frac{1}{2} \varepsilon_n$ in **Theorem 1**, we can see that the neighborhood of $\{c_n\}$ contains at least one point which belongs to $\mathbf{S}_1$. q. e. d.

References

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